













# **INTERMEDIATE TRIGONOMETRY**

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# INTERMEDIATE TRIGONOMETRY

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## PREFACE TO THE FIRST EDITION

**T**HIS book, as its name indicates, is meant to be a textbook for the Intermediate students of Indian Universities, especially the University of Calcutta. Regarding the subject-matter, we have tried to make the exposition clear and concise, without going into unnecessary details. A good number of examples have been worked out by way of illustrations, and examples set have been carefully selected.

Important formulæ and results have been given at the beginning of the book for reference. Calcutta University questions of recent years are given at the end, to give the students an idea of the standard of the examination.

It is hoped that the book will meet the requirements of those for whom it is intended and we shall deem our labours amply rewarded if the students find the book useful to them.

The book had to be hurried through the press practically within the period of a fortnight, and we must thank the authorities and officers of the K. P. Basu Printing Works, Calcutta, who, in spite of their various preoccupations had the kindness to complete the printing in such a short period of time.

Any criticism, correction and suggestion towards improvement will be thankfully received.

B. C. D.

B. N. M.



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### GREEK LETTERS USED IN THE BOOK

$\alpha$ (Alpha)	$\theta$ (Thetā)
$\beta$ (Bētā)	$\pi$ (Pai)
$\gamma$ (Gamma)	$\phi$ (Phai)
$\delta$ (Deltā)	$\psi$ (Psi)

$\Delta$  (Deltā)

**Note.** The notation *C. U.* used at the end of any example means that the example was set in Intermediate Examination of the Calcutta University.

## TRIGONOMETRY SYLLABUS FOR INTERMEDIATE EXAMINATIONS

Measurement of angles. Trigonometric ratios and their graphs. Elementary Trigonometrical formulæ. Trigonometric ratios of associated angles. Summation Theorems. Transformation of products and sums. Multiple and sub-multiple angles. Trigonometric Equations and General Values. Inverse Circular Functions. Relation between sides and angles of a triangle. In-radius, Circum-radius and Area of a triangle. In-radius, Circum-radius and Area of a triangle. Partical solutions using log tables. Simple problems in heights and distances.

## TRIGONOMETRY SYLLABUS FOR PRE-UNIVERSITY AND ENTRANCES

Measurement of angles—Sexagesimal and circular measures. Definition of trigonometrical ratios, their mutual relations. Deductions of the values of the trigonometrical ratios of  $0^\circ$ ,  $30^\circ$ ,  $45^\circ$ ,  $60^\circ$ ,  $90^\circ$ . Trigonometrical ratios of associated angle. Addition and subtraction formulæ. Transformation of products and sums of trigonometrical ratios. Multiple and sub-multiple angles (simple cases). General values, Solution of trigonometrical equations. Inverse circular functions. Trigonometrical identities. Relation between sides and angles of a triangle, area, in-radius and circum-radius of a triangle. Solution of triangles with the use of log tables. Graphs of simple trigonometrical functions. Simple problems of heights and distances.

# TRIGONOMETRY SYLLABUS FOR HIGHER SECONDARY COURSE

## Class IX

Measurement of angles in degrees, minutes, seconds and in radians. Definition of trigonometrical ratios of an acute angle. Trigonometrical ratios of the standard angles— $0^\circ$ ,  $30^\circ$ ,  $45^\circ$ ,  $60^\circ$ ,  $90^\circ$ , (undefined values such as  $\tan 90^\circ$ ,  $\cot 0^\circ$  to be excluded). Simple identities connecting the ratios of an angle immediately derivable from a right-angled triangle. Trigonometrical ratios of complementary angles.

Easy problems on heights and distances reducible to the solution of right-angled triangles involving the standard angles above.

## Class X

Trigonometrical ratios of any angle ; Trigonometrical ratios of angles associated with a given angle ; Addition and subtraction formulæ ; Transformation of products and sums ; Multiple and sub-multiple angles.

## Class XI

Graphs of simple trigonometric functions.

Trigonometric equations and general values ; Inverse Circular Functions.

Relation between sides and angles of a triangle ; In-radius, circum-radius and area of a triangle ; Practical solution of a triangle with the help of logarithms ; Simple problems of heights and distances.

## IMPORTANT FORMULÆ AND RESULTS

**I.** A radian =  $57^{\circ} 17' 44.8''$  nearly.

1 degree =  $.01745$  radians nearly.

2 right angles =  $180^{\circ} = \pi$  radians.

$\pi = \frac{22}{7} = 3.1416$  approximately.

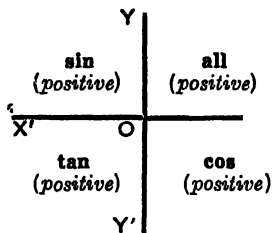
Radian measure of an angle at the centre of a circle

$$= \frac{\text{subtending arc}}{\text{radius}}.$$

$$\text{II. } \left. \begin{array}{l} \sin^2 \theta + \cos^2 \theta = 1 ; \\ \sec^2 \theta = 1 + \tan^2 \theta ; \\ \operatorname{cosec}^2 \theta = 1 + \cot^2 \theta. \end{array} \right\} \begin{array}{l} \frac{\sin \theta}{\cos \theta} = \tan \theta. \\ \frac{\cos \theta}{\sin \theta} = \cot \theta. \end{array}$$

$$\begin{array}{lll} \text{III. } \sin 0^{\circ} = 0 ; & \cos 0^{\circ} = 1 ; & \tan 0^{\circ} = 0. \\ \sin 30^{\circ} = \frac{1}{2} ; & \cos 30^{\circ} = \frac{\sqrt{3}}{2} ; & \tan 30^{\circ} = \frac{1}{\sqrt{3}}. \\ \sin 45^{\circ} = \frac{1}{\sqrt{2}} ; & \cos 45^{\circ} = \frac{1}{\sqrt{2}} ; & \tan 45^{\circ} = 1. \\ \sin 60^{\circ} = \frac{\sqrt{3}}{2} ; & \cos 60^{\circ} = \frac{1}{2} ; & \tan 60^{\circ} = \sqrt{3}. \\ \sin 90^{\circ} = 1 ; & \cos 90^{\circ} = 0 ; & \tan 90^{\circ} = \infty. \\ \sin 15^{\circ} = \frac{\sqrt{3}-1}{2\sqrt{2}} ; & \cos 15^{\circ} = \frac{\sqrt{3}+1}{2\sqrt{2}} ; & \tan 15^{\circ} = 2 - \sqrt{3}. \\ \sin 75^{\circ} = \frac{\sqrt{3}+1}{2\sqrt{2}} ; & \cos 75^{\circ} = \frac{\sqrt{3}-1}{2\sqrt{2}} ; & \tan 75^{\circ} = 2 + \sqrt{3}. \\ \sin 18^{\circ} = \frac{1}{4}(\sqrt{5}-1) ; & \cos 36^{\circ} = \frac{1}{4}(\sqrt{5}+1). \\ \sin 120^{\circ} = \frac{\sqrt{3}}{2} ; & \cos 120^{\circ} = -\frac{1}{2}. \\ \sin 180^{\circ} = 0 ; & \cos 180^{\circ} = -1 ; & \tan 180^{\circ} = 0. \\ \sin 270^{\circ} = -1 ; & \cos 270^{\circ} = 0 ; & \tan 270^{\circ} = \infty. \\ \sin 360^{\circ} = 0 ; & \cos 360^{\circ} = 1 & \tan 360^{\circ} = 0. \end{array}$$

- IV.  $\sin(-\theta) = -\sin \theta$ ;  $\cos(-\theta) = \cos \theta$ ;  $\tan(-\theta) = -\tan \theta$ .  
 $\sin(90^\circ - \theta) = \cos \theta$ ;  $\sin(90^\circ + \theta) = \cos \theta$ .  
 $\cos(90^\circ - \theta) = \sin \theta$ ;  $\cos(90^\circ + \theta) = -\sin \theta$ .  
 $\tan(90^\circ - \theta) = \cot \theta$ ;  $\tan(90^\circ + \theta) = -\cot \theta$ .  
 $\sin(180^\circ - \theta) = \sin \theta$ ;  $\sin(180^\circ + \theta) = -\sin \theta$ .  
 $\cos(180^\circ - \theta) = -\cos \theta$ ;  $\cos(180^\circ + \theta) = -\cos \theta$ .  
 $\tan(180^\circ - \theta) = -\tan \theta$ ;  $\tan(180^\circ + \theta) = \tan \theta$ .



- V.  $\sin(A+B) = \sin A \cos B + \cos A \sin B$   
 $\sin(A-B) = \sin A \cos B - \cos A \sin B$   
 $\cos(A+B) = \cos A \cos B - \sin A \sin B$   
 $\cos(A-B) = \cos A \cos B + \sin A \sin B$   
 $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$   
 $\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$   
 $\tan(A+B+C)$   
 $= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan B \tan C - \tan C \tan A - \tan A \tan B}$ .
- VI.  $2 \sin A \cos B = \sin(A+B) + \sin(A-B)$   
 $2 \cos A \sin B = \sin(A+B) - \sin(A-B)$   
 $2 \cos A \cos B = \cos(A+B) + \cos(A-B)$   
 $2 \sin A \sin B = \cos(A-B) - \cos(A+B)$ .
- VII.  $\sin C + \sin D = 2 \sin \frac{C+D}{2} \cos \frac{C-D}{2}$   
 $\sin C - \sin D = 2 \cos \frac{C+D}{2} \sin \frac{C-D}{2}$

$$\cos C + \cos D = 2 \cos \frac{C+D}{2} \cos \frac{C-D}{2}$$

$$\cos C - \cos D = 2 \sin \frac{C+D}{2} \sin \frac{D-C}{2}.$$

**VIII.**  $\sin 2A = 2 \sin A \cos A$

$$\cos 2A = \cos^2 A - \sin^2 A = 1 - 2 \sin^2 A = 2 \cos^2 A - 1$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

$$\sin 2A = \frac{2 \tan A}{1 + \tan^2 A}; \quad \cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

$$\left. \begin{aligned} 1 - \cos 2A &= 2 \sin^2 A \\ 1 + \cos 2A &= 2 \cos^2 A \end{aligned} \right\}$$

$$\tan^2 A = \frac{1 - \cos 2A}{1 + \cos 2A}.$$

**IX.**  $\sin 3A = 3 \sin A - 4 \sin^3 A$

$$\cos 3A = 4 \cos^3 A - 3 \cos A$$

$$\tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}.$$

**X.**  $\sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}$

$$\cos \theta = \cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2} = 2 \cos^2 \frac{\theta}{2} - 1 = 1 - 2 \sin^2 \frac{\theta}{2}$$

$$\tan \theta = \frac{2 \tan \frac{\theta}{2}}{1 - \tan^2 \frac{\theta}{2}}$$

$$\sin \theta = \frac{2 \tan \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}}; \quad \cos \theta = \frac{1 - \tan^2 \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}}$$

$$1 - \cos \theta = 2 \sin^2 \frac{\theta}{2}$$

$$1 + \cos \theta = 2 \cos^2 \frac{\theta}{2}$$

$$\frac{1 - \cos \theta}{1 + \cos \theta} = \tan^2 \frac{\theta}{2}.$$

**XI.** If  $\sin \theta = \sin a$ , then  $\theta = n\pi = (-1)^n a$ .

If  $\cos \theta = \cos a$ , then  $\theta = 2n\pi \pm a$ .

If  $\tan \theta = \tan a$ , then  $\theta = n\pi + a$ .

If  $\sin \theta = 0$ , or,  $\tan \theta = 0$ ,  $\theta = n\pi$ .

If  $\cos \theta = 0$ , or,  $\cot \theta = 0$ ,  $\theta = (2n+1)\frac{\pi}{2}$ .

If  $\sin \theta = 1$ ,  $\theta = (4m+1)\frac{\pi}{2}$ ; if  $\sin \theta = -1$ ,  $\theta = (4m-1)\frac{\pi}{2}$ .

If  $\cos \theta = 1$ ,  $\theta = 2m\pi$ ; if  $\cos \theta = -1$ ,  $\theta = (2m+1)\pi$ .

**XII.**  $\sin^{-1}x + \cos^{-1}x = \frac{1}{2}\pi$

$$\tan^{-1}x + \cot^{-1}x = \frac{1}{2}\pi$$

$$\sec^{-1}x + \operatorname{cosec}^{-1}x = \frac{1}{2}\pi$$

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1} \frac{x+y}{1-xy}$$

$$\tan^{-1}x - \tan^{-1}y = \tan^{-1} \frac{x-y}{1+xy}$$

$$\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1} \frac{x+y+z-xyz}{1-yz-zx-xy}$$

$$\sin^{-1}x \pm \sin^{-1}y = \sin^{-1}\{x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\}$$

$$\cos^{-1}x \pm \cos^{-1}y = \cos^{-1}\{xy \mp \sqrt{1-x^2} \cdot \sqrt{1-y^2}\}.$$

**XIII.**  $\log_a mn = \log_a m + \log_a n$

$$\log_a \frac{m}{n} = \log_a m - \log_a n; \log_a m^n = n \log_a m;$$

$$\log_a m = \log_b m \times \log_a b; \log_a 1 = 0; \log_a a = 1.$$

**XIV.**  $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc};$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca};$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab};$$

$$a = b \cos C + c \cos B.$$

$$b = c \cos A + a \cos C.$$

$$c = a \cos B + b \cos A.$$

$$\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

$$\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$$

$$\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$\sin A = \frac{2}{bc} \sqrt{s(s-a)(s-b)(s-c)} = \frac{2\Delta}{bc}$$

$$\sin B = \frac{2}{ca} \sqrt{s(s-a)(s-b)(s-c)} = \frac{2\Delta}{ca}$$

$$\sin C = \frac{2}{ab} \sqrt{s(s-a)(s-b)(s-c)} = \frac{2\Delta}{ab}$$

$$\begin{aligned} \Delta &= \frac{1}{2}bc \sin A = \frac{1}{2}ca \sin B = \frac{1}{2}ab \sin C \\ &= \sqrt{s(s-a)(s-b)(s-c)}, \text{ where } 2s = a + b + c \\ &= \frac{abc}{4R}. \end{aligned}$$

$$R = \frac{a}{2 \sin A} = \frac{b}{2 \sin B} = \frac{c}{2 \sin C} = \frac{abc}{4\Delta}.$$

$$\begin{aligned} r &= \frac{\Delta}{s} = 4R \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C \\ &= (s-a) \tan \frac{1}{2}A = (s-b) \tan \frac{1}{2}B = (s-c) \tan \frac{1}{2}C. \end{aligned}$$

$$\begin{aligned} r_1 &= \frac{\Delta}{s-a} = 4R \sin \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C \\ &= s \tan \frac{1}{2}A. \end{aligned}$$

$$\begin{aligned} r_2 &= \frac{\Delta}{s-b} = 4R \cos \frac{1}{2}A \sin \frac{1}{2}B \cos \frac{1}{2}C \\ &= s \tan \frac{1}{2}B. \end{aligned}$$

$$\begin{aligned} r_3 &= \frac{\Delta}{s-c} = 4R \cos \frac{1}{2}A \cos \frac{1}{2}B \sin \frac{1}{2}C \\ &= s \tan \frac{1}{2}C. \end{aligned}$$



### IMPORTANT RESULTS

1. If  $A + B + C = \pi$ , then

$$(i) \sin A + \sin B + \sin C = 4 \cos \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C.$$

$$(ii) \cos A + \cos B + \cos C = 1 + 4 \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C.$$

$$(iii) \tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

$$(iv) \sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C.$$

$$(v) \cos 2A + \cos 2B + \cos 2C \\ = -4 \cos A \cos B \cos C - 1.$$

$$(vi) \cos^2 A + \cos^2 B + \cos^2 C + 2 \cos A \cos B \cos C = 1.$$

$$(vii) \cot B \cot C + \cot C \cot A + \cot A \cot B = 1.$$

$$(viii) \sin \frac{A}{2} + \sin \frac{B}{2} + \sin \frac{C}{2} \\ = 1 + 4 \sin \frac{B+C}{4} \sin \frac{C+A}{4} \sin \frac{A+B}{4}.$$

$$(ix) \cos \frac{A}{2} + \cos \frac{B}{2} + \cos \frac{C}{2} \\ = 4 \cos \frac{B+C}{4} \cos \frac{C+A}{4} \cos \frac{A+B}{4}.$$

$$(x) \tan \frac{B}{2} \tan \frac{C}{2} + \tan \frac{C}{2} \tan \frac{A}{2} + \tan \frac{A}{2} \tan \frac{B}{2} = 1.$$

$$(xi) \cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}.$$

$$2. \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1; \lim_{\theta \rightarrow 0} \cos \theta = 1; \lim_{\theta \rightarrow 0} \frac{\tan \theta}{\theta} = 1.$$

3. Area of a circle of radius  $r = \pi r^2$ .

Perimeter of a circle of radius  $r = 2\pi r$ .

—

# TRIGONOMETRY



## CHAPTER I

### MEASUREMENT OF ANGLES

1. **TRIGONOMETRY**, as indicated by its very name, originally meant a subject which dealt with the methods of measurement of triangles. At present its scope has widened, and now it means a subject which deals with the measurements relating to any angle, not necessarily an angle of a triangle.

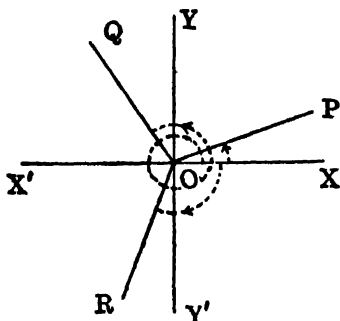
#### 2. Angles in Trigonometry.

In Geometry, angles are supposed to be formed by the intersection of two straight lines and are always restricted to lie between  $0^\circ$  and  $360^\circ$ , being acute, obtuse or reflex. Moreover, they are always positive, negative angles having no meaning. In Trigonometry however, the idea of an angle is much more general.

An angle in Trigonometry is supposed to be formed by the revolution of a straight line which starts from an initial position coinciding with one arm, and traces out the angle by its revolution about one extremity until it reaches the final position coinciding with the other arm.

For instance, the angle  $XOP$  is formed by the revolution of a line which starts from the initial position  $OX$ , and revolving in the anti-clockwise direction, traces out the angle  $XOP$  which is acute. The same line again, starting from  $OX$  and revolving in the anti-clockwise direction may make a complete revolution and further move up to the position  $OQ$ . The angle formed in this case is more than

five right angles. Now revolutions may be clockwise or anti-clockwise. *It is conventional to consider angles formed by the anti-clockwise revolution of the revolving line to be positive.* Angles formed by clockwise revolutions of the



revolving line will then be considered *negative angles*. For example, the angle  $XOR$  measured in the clockwise direction from the initial position  $OX$  is a negative angle.

Thus, *angles in Trigonometry may be of any magnitude and may be positive as well as negative.*

$OX$  being the initial position of the revolving line, produce  $XO$  to  $X'$ , and let  $YOY'$  be the perpendicular line. The whole plane is thus divided into four quadrants, the first being  $XOY$ , the second  $YOX'$ , the third  $X'OY'$  and the fourth  $Y'OX$ . If we contemplate an angle say  $+920^\circ$  to be traced out by the revolving line, the line must have completed two complete revolutions, thereby describing  $2 \times 360^\circ = 720^\circ$ , and have further traced out an angle  $200^\circ$ , so that the final position of the revolving line is in the third quadrant. Similarly, if we consider an angle  $-1354^\circ$ , the final position of the revolving line is in the first quadrant, for  $-1354^\circ = -360^\circ \times 3 - 274^\circ$ .

It should be noted that if two angles differ by complete multiples of  $360^\circ$ , the starting line being the same, the final

position of the revolving line will be coincident for the two angles. For example, the angles  $255^\circ$  and  $-105^\circ$  will have the final positions of the revolving line same, if both start from the same initial position.

### 3. Units of measurement of angles.

We should now define the different systems of units used for the measurement of angles. In defining a unit however, a standard angle, which has no reference to any particular system of unit, should form the basis, and such a standard angle is a right angle. A right angle is defined in books on Geometry to be an angle which any straight line standing on another makes with it, when the two adjacent angles formed are equal to one another. A right angle is always the same everywhere, and it thus forms a suitable basis to start with, in defining the different systems of measurement of angles.

There are three systems of units used in Trigonometry for measurement of angles, *viz.*,

(i) Sexagesimal unit.

(ii) Centesimal unit.

(iii) Circular unit.

**Sexagesimal\* System.** In this system, a right angle is divided into 90 equal parts, each being called a *degree*. A degree is again divided into 60 sexagesimal *minutes*, and each minute is further subdivided into 60 sexagesimal *seconds*, so that

1 rt. angle =  $90^\circ$  (degrees)

1° = 60' (sexagesimal minutes)

1' = 60" (sexagesimal seconds)

---

\* So called, since the subdivisions are mostly by sixtieth parts. It is also called the *Common* or the *English System*.

**Centesimal† System.** In this system, the subdivisions of a right angle are as follows :

$$1 \text{ rt. angle} = 100^g \text{ (grades)}$$

$$1^g = 100' \text{ (centesimal minutes)}$$

$$1' = 100'' \text{ (centesimal seconds).}$$

**Note.** It may be noted that  $1'$  (centesimal minute) is not the same as  $1'$  (sexagesimal minute), the former being  $\frac{1}{100 \times 100}$  of a right angle and the latter being  $\frac{1}{90 \times 60}$  of a right angle, so that the first is  $\frac{27}{50}$ th part of the second. Similarly,  $1''$  is less than  $1''$ , being only  $\frac{81}{2500}$ th part of it.

The connection between the two systems of units may be effected through a right angle, remembering that 1 right angle  $= 90^\circ = 100^g$ , so that  $9^\circ = 10^g$ . Any angle in the first system may be reduced to degrees, and then multiplied by  $\frac{10}{9}$  will be reduced to grades. Similarly, an angle in the second system may be changed to the first.

We shall presently deal with the third system, namely the circular system.

**4. Theorem.** *In all circles, the circumference bears a constant ratio to its diameter.*

Take any two circles of any radii, and place them with a common centre  $O$ . In one, let  $ABCD\dots$  be an inscribed regular polygon of  $n$  sides. Let  $A', B', C', \dots$  be the points of intersection of the radii  $OA, OB, OC, \dots$  with the other circle. It is easily seen that  $A'B'C'\dots$  is also a regular polygon of  $n$  sides, inscribed in the second circle. Now  $OA = OB$  as also  $OA' = OB'$ , so that in the triangles  $OAB, OA'B'$ ,

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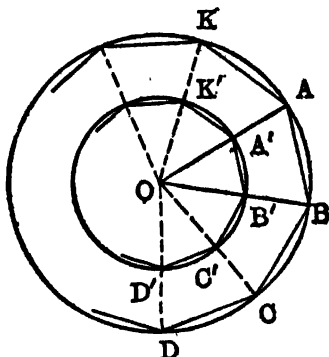
† So called because the subdivisions are by hundredths. It is also called the *French System*.

## MEASUREMENT OF ANGLES

$OA : OA' = OB : OB'$ , and angle  $O$  is common. The two triangles are therefore similar. Hence,  $AB : A'B' = OA : OA'$ .

Thus,

$$\frac{\text{perimeter of polygon } ABCD\dots}{\text{perimeter of polygon } A'B'C'D'\dots} = \frac{n \cdot AB}{n \cdot A'B'} = \frac{OA}{OA'}.$$



This being true, whatever the number of sides  $n$  may be, making  $n$  infinitely large, the perimeters of the polygons can be made practically coincident with the circumferences of the corresponding circles, and thus we deduce that

$$\begin{aligned} \frac{\text{circumference of the circle } ABCD\dots}{\text{circumference of the circle } A'B'C'D'\dots} &= \frac{OA}{OA'} \\ \therefore &= \frac{\text{radius of circle } ABC\dots}{\text{radius of circle } A'B'C'\dots} \end{aligned}$$

Thus circumference of any circle : its radius is the same for all circles. As diameter is twice the radius, we deduce that the circumference of any circle bears a constant ratio to its diameter.

This constant ratio is denoted by the Greek letter  $\pi$ . Its actual value has been determined by methods which are outside the scope of the present book, by some mathematicians

to more than 500 places of decimals. An approximate value commonly used is  $\frac{22}{7}$ . A more accurate value is  $\frac{355}{113}$ .

Expressed in decimal, the value is nearly  $3.14159\dots$

Hence, if  $r$  be the radius of a circle,  $d$  its diameter,  
the circumference  $= \pi d = 2\pi r$ ,

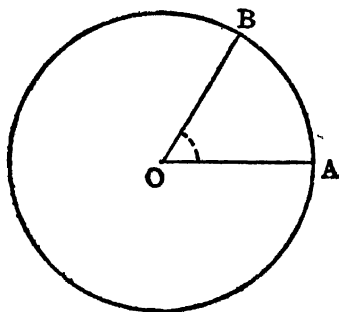
where  $\pi = 3.14159\dots = \frac{22}{7}$  roughly.

### 5. Circular Unit or Radian Measure.

In any circle, if we take an arc whose length is equal to the radius of the circle, the angle which this arc subtends at the centre, is called a *radian*, and is written as  $1^c$ .

We shall now show that with reference to whichever circle it may be defined, a radian is a constant angle, and hence it may be used as a suitable unit for measurement of angles, which is known as the circular unit.

**Theorem I.** *A radian is a constant angle.*



Let  $AB$  be an arc of any circle with centre  $O$ , whose length is equal to its radius  $OA$ . By definition,  $\angle AOB = 1$  radian. Since angles at the centre of a circle are proportional to the arcs which subtend them, and the whole angle

round  $O$  subtended by the complete circumference being known from Geometry to be 4 right angles, we get

$$\frac{\angle AOB}{4 \text{ right angles}} = \frac{\text{arc } AB}{\text{whole circumference}} = \frac{\text{radius}}{\text{circumference}},$$

$$\text{i.e., } \frac{1 \text{ radian}}{4 \text{ rt. } \angle} = \frac{r}{2\pi r} = \frac{1}{2\pi}, r \text{ being the radius.}$$

$$\text{Hence, } 1 \text{ radian} = \frac{2}{\pi} \text{ rt. angle.}$$

a radian is a constant angle ( $\pi$  being constant).

**Note.** We thus see that whatever be the radius of the circle with reference to which a radian is defined, its magnitude is the same.

From above,  $\pi$  radians =  $180^\circ$ .

$$\therefore 1 \text{ radian} = \frac{180}{\pi} = \frac{180}{3.14159} = 57.29577 \text{ degrees}$$

$$= 57^\circ 17' 44.8'' \text{ nearly,}$$

$$\therefore 1 \text{ degree} = .0174533 \text{ radians nearly.}$$

In higher mathematics so far as theoretical investigations are concerned, as a matter of convenience, angles are usually measured in the circular unit, *i.e.*, in radians. In this connection we may state the following theorem:

**Theorem II.** *The measure of any angle in radians is expressed by the ratio of the arc of any circle subtending that angle at its centre, to the radius.*

Let  $XOP$  be any angle.

With centre  $O$  and any radius  $OA$  draw a circle, and let  $AQ$  be the arc which subtends the angle  $XOP$  at the centre  $O$ . Let  $AB$  be the arc whose length is equal to the radius  $AO$ , so that, by definition,  $\angle AOB$  is one radian.

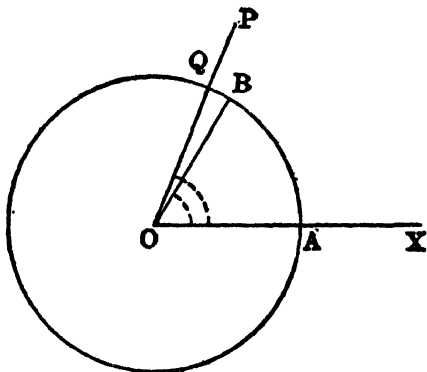
Now from Geometry, angles at the centre of a circle are proportional to the arcs which subtend them.



$$\text{Hence, } \frac{\angle XOP}{\angle AOB} = \frac{\text{arc } AQ}{\text{arc } AB} = \frac{\text{arc } AQ}{\text{radius } OA},$$

$$\text{or, } \frac{\angle XOP}{1 \text{ radian}} = \frac{\text{arc } AQ}{\text{radius } OA},$$

$$\text{i.e., } \angle XOP = \frac{\text{arc } AQ}{\text{radius } OA} \text{ of a radian.}$$



Thus, if  $\theta$  be the radian-measure of the  $\angle XOP$ ,  $s$  be the length of the arc  $AQ$ , and  $r$  the radius of the circle, then

$$\theta = \frac{s}{r} \text{ or, } s = r\theta.$$

**Note.** In higher mathematics, when an angle is expressed in radian-measure, the unit is generally implied and not expressed, so that, when the measure of an angle is given without the unit being mentioned, we should always understand it to be in radians. For example, 'an angle is  $\frac{\pi}{2}$ ' means that the angle is  $\frac{\pi}{2}$  radians, which converted to degrees is  $90^\circ$  i.e., one right angle.

6. In working out examples, relations between the three systems of units should be carefully remembered, namely

$$1 \text{ rt. } \angle = 90^\circ = 100^\circ = \frac{\pi}{2} \text{ radians,}$$

$$\text{whence, } \pi^\circ = 180^\circ.$$

**Ex. 1.** *Express*

(i)  $63^{\circ} 22' 40.8''$  in centesimal measure  
and (ii)  $203^{\circ} 58' 73''$  in radians.

$$\begin{aligned}\text{Here, (i) } 63^{\circ} 22' 40.8'' &= 63\frac{1462}{1000} \text{ deg.} = \frac{317331}{500} \times \frac{1}{100} \text{ rt. } \angle \\ &= \frac{317331}{500} \times \frac{1}{100} \times 100 \text{ grades} = \frac{317331}{5} \text{ grades} \\ &= 70^{\circ} 42' .\end{aligned}$$

$$\begin{aligned}\text{(ii) } 203^{\circ} 58' 73'' &= 203.5873 \text{ grades} \\ &= 2.035873 \text{ rt. } \angle = 2.035873 \times \frac{\pi}{2} \text{ radians} \\ &= 1.0179365\pi \text{ radians.}\end{aligned}$$

**Ex. 2.** *Two angles of a triangle are  $72^{\circ} 53' 51''$ , and  $41^{\circ} 22' 50''$  respectively. Find the third angle in radians.*

$$\begin{aligned}41^{\circ} 22' 50'' &= 41.2250 \text{ grades} \\ &= \frac{41.225 \times 9}{10} \text{ degrees } [ 9^{\circ} = 10^{\circ} ] \\ &= 37.1025 \text{ degrees} \\ &= 37^{\circ} 6' 9'' .\end{aligned}$$

The sum of the two given angles is therefore

$$72^{\circ} 53' 51'' + 37^{\circ} 6' 9'' = 110^{\circ} .$$

The sum of the three angles of a triangle being  $180^{\circ}$ , the third angle is

$$\begin{aligned}180^{\circ} - 110^{\circ} &= 70^{\circ} = 70^{\circ} \times \frac{\pi}{180} \text{ radians } [ \pi^{\circ} = 180^{\circ} ] \\ &= \frac{7\pi}{18} \text{ radians.}\end{aligned}$$

**Ex. 3.** *Divide  $\frac{\pi}{4}$  radians into two parts such that the number of sexagesimal minutes in one may be to the number of centesimal seconds in the other part as 27 : 2500.*

$$\text{We have } \frac{\pi}{4} \text{ radians} = \frac{\pi}{4} \times \frac{2}{\pi} \text{ rt. } \angle = \frac{1}{2} \text{ rt. } \angle .$$

Let  $x$  be the number of centesimal seconds in the second part, so that  $\frac{27}{2500}x$  is the number of sexagesimal minutes in the first part.

$$\text{Now, } x'' = \frac{x}{100 \times 100 \times 100} \text{ rt. } \angle$$

$$\text{and } \frac{27}{2500}x' = \frac{27x}{2500 \times 60 \times 90} \text{ rt. } \angle = \frac{x}{500000} \text{ rt. } \angle,$$

$$\therefore \frac{x}{1000000} + \frac{x}{500000} = \frac{1}{2},$$

$$\text{whence } x = \frac{500000}{3}.$$

$$\begin{aligned} \text{Thus, second part is } \frac{500000''}{3} &= \frac{500000}{3 \times 100 \times 100 \times 100} \text{ rt. } \angle \\ &= \frac{1}{3} \text{ rt. } \angle = 15^\circ, \text{ and as the sum of the two parts is } \\ &\frac{1}{2} \text{ rt. } \angle \text{ i.e., } 45^\circ, \text{ the first part is } 30^\circ. \end{aligned}$$

The two parts are therefore  $30^\circ$  and  $15^\circ$ .

**Ex. 4.** *The angles of a quadrilateral are in A.P., and the number of grades in the least angle is to the number of radians in the greatest as  $100 : \pi$ . Find the angles in degrees.*

Let the angles, expressed in degrees, be  $\alpha$ ,  $\alpha + \beta$ ,  $\alpha + 2\beta$  and  $\alpha + 3\beta$  respectively. Then

$$\alpha + \alpha + \beta + \alpha + 2\beta + \alpha + 3\beta = 360, \quad \dots (1)$$

$$\text{i.e., } 2\alpha + 3\beta = 180.$$

$$\text{Again the least angle, } \alpha^\circ = \frac{100}{\pi} \alpha^\circ$$

$$\text{and the greatest angle, } (\alpha + 3\beta)^\circ = (\alpha + 3\beta) \frac{\pi^\circ}{180},$$

and so from the given condition,

$$\frac{10}{9} \alpha / (\alpha + 3\beta) \frac{\pi}{180} = 100/\pi,$$

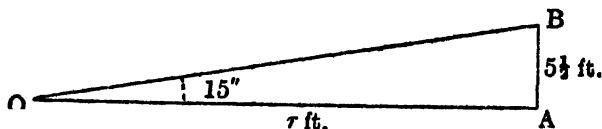
$$\text{or, } \frac{2\alpha}{\alpha + 3\beta} = 1, \text{ whence } \alpha = 3\beta.$$

$\therefore$  using (1),  $3\alpha = 180$ , or,  $\alpha = 60$  and  $\beta = \frac{\alpha}{3} = 20$ .

Thus the angles are

$60^\circ, 80^\circ, 100^\circ$  and  $120^\circ$ .

**Ex. 5.** *At what distance does a man,  $5\frac{1}{2}$  ft. in height subtend an angle of  $15''$ ?*



$AB$  being the man subtending an angle  $15''$  at  $O$ , let  $OA$  be  $r$  ft.

As the angle  $AOB$  is very small, so that  $AB$  is very small compared to  $AO$ , we may assume the small length  $AB$  to be practically a small arc of a circle whose centre is  $O$ . Now the measure of an angle in radians is the ratio of the arc which subtends it at the centre to the radius.

[ See Art. 5 ]

$$\therefore \frac{15}{60 \times 60} \times \frac{\pi}{180} = \frac{5\frac{1}{2}}{r},$$

$$\text{or, } r = \frac{11}{2} \times \frac{180 \times 60 \times 60}{15 \times \pi} \text{ ft.}$$

$$= \frac{11}{2} \times \frac{180 \times 60 \times 60 \times 7}{15 \times 22} \times \frac{1}{3 \times 1760} \text{ miles approx.}$$

$$= 14.32 \text{ miles nearly.}$$

### Examples I

1. Indicate the final position of a revolving line which has traced out the angle

(i)  $1122^\circ$ ;                      (ii)  $-810^\circ 29'$ ;

(iii)  $-617^\circ 51' 5''$ ;      (iv)  $\frac{18\pi}{5}$  radians.

2. Express (i)  $55^{\circ} 12' 36''$  in centesimal measure ;  
(ii)  $195^{\circ} 35' 24''$  in degrees, minutes, and secs.
3. How many radians are there in (i)  $50^{\circ} 75' 50''$  ;  
(ii)  $18^{\circ} 33' 45''$  ?
4. Express in each system of angular measurement the angle between the minute-hand and the hour-hand of a clock at quarter to twelve.
5. If  $x^{\circ}$  be taken as the unit angle, and the angles  $600^{\circ}$  and  $16^{\circ}$  expressed in that unit be  $\alpha$  and  $\beta$  respectively, find the relation between  $\alpha$  and  $\beta$ .
6. The difference of two angles is  $1^{\circ}$  ; the circular measure of their sum is 1 ; find the circular measure of the smaller angle.
7. Two angles are in the ratio 2 : 3, and the difference of their measure in grades and in degrees respectively is  $2\frac{1}{2}$  ; find the angles in degrees.
8. An angle is the excess of  $D^{\circ} M'$  over  $G^{\circ} m'$ . Find the ratio of this angle to a right angle.
9. The circular measure of a certain angle is equal to the ratio of the number of degrees in it to the number of centesimal minutes ; find the magnitude of the angle in degrees.
10. With two units of angular measurements differing by  $10^{\circ}$ , the measure of an angle are as 3 : 2 ; determine the units.
11. If an angle standing upon an arc of length ' $l$ ' at the centre of a circle of radius ' $r$ ' be taken as unit, and three angles  $D^{\circ}$ ,  $G^{\circ}$ , and  $C$  circular units expressed in that unit be  $x$ ,  $y$ ,  $z$  respectively, show that

$$x : y : z = \frac{D\pi}{18} : \frac{G\pi}{20} : 100.$$

12. Three angles are in G.P. The number of grades in the greatest angle is to the number of circular units in the least as 800 to  $\pi$ , and the sum of the three angles is  $126^\circ$ . Find the angles in grades.

13. Divide  $54^\circ$  in three parts, such that the circular measure of the first exceeds that of the second by  $\frac{\pi}{10}$ , and the sum of the second and third is 30 grades.

14. Find at what times between 7 and 8 o'clock the angle between the two hands of a clock is (i)  $60^\circ$ , (ii)  $155^\circ$ .

15. The angles of a triangle are in A.P., and the number of radians in the greatest is to the number of grades in the least as  $\pi : 40$ . Find the angles in degrees.

16. In each of two triangles the angles are in G.P.; the least angle of one of them is three times the least angle in the other, and the sum of the greatest angles is  $240^\circ$ . Find the circular measure of the angles.

17. One angle of a quadrilateral is  $\frac{2}{3}$  of another and the two other angles are  $66\frac{2}{3}$  grades and  $\frac{3}{4}\pi$  radians. Express the angles in degrees.

18. The angles of a polygon (which has no reflex angle) are in A.P. The least angle is  $\frac{2}{3}\pi$  radians and the common difference is  $5^\circ$ . Find the number of sides.

19. The number of sides of two regular polygons are as  $m : n$  and the number of degrees in an angle of the first is to the number of grades in an angle of the second as  $p : q$ . Determine the number of sides in each polygon.

20. An arc of  $50^\circ$  in one circle equals one of  $60^\circ$  in another; find the radian-measure of an angle subtended at the centre of the first circle by an arc equal to the radius of the second.

21. Two regular figures are such that the number of degrees in an angle of one is to the number of degrees in an angle of the other as the number of sides in the first is to the number of sides in the second. The sum of the number of sides of the two figures being 9, determine the number of sides of each.

22. The wheel of a railway carriage is 4 ft. in diameter and makes 6 revolutions in a second; how fast is the train going?

23. The earth revolves round the sun in a circular orbit of radius 92700000 miles once a year. Find its velocity in miles per hour. If the apparent angular diameter of the sun observed from the earth be  $32'$ , find also the linear radius of the sun.

24. A tower subtends an angle of  $10'$  when the observer is at a distance of 6 miles; find its height.

25. Find the radius of the earth, if an angle of  $1^\circ$  is subtended at its centre by an arc joining two places on it distant  $69.1$  miles.

26. A horse is tied to a post by a rope 27 feet long. If the horse moves along the circumference of a circle always keeping the rope tight, find how far the horse will have gone when the rope has traced out an angle of  $70^\circ$ .  
( $\pi = \frac{22}{7}$ )

27. A man running along a circular track at the rate of 10 miles per hour, traverses in 36 seconds, an arc which subtends  $56^\circ$  at the centre. Find the diameter of the circle.  
( $\pi = \frac{22}{7}$ )

28. An arc of  $30^\circ$  in one circle is double an arc in a second circle, the radius of which is three times the radius of the first. Show that the arc of the second circle subtends  $5^\circ$  at its centre.

# CHAPTER II

## TRIGONOMETRICAL RATIOS\*

### 7. Trigonometrical ratios defined.

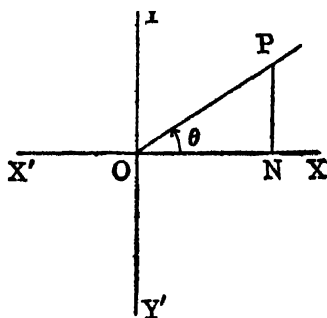


Fig. 1

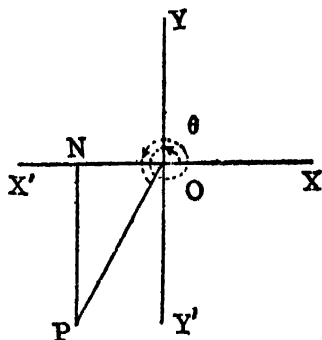


Fig. 2

Let  $\theta$  be the measure of an angle  $XOP$  which may be supposed to be traced out by a revolving line starting from the initial position  $OX$ . From any point  $P$  on its other arm, draw a perpendicular  $PN$  on  $OX$  (produced if necessary, as in the second figure). A right-angled triangle is thereby formed. The trigonometrical ratios of the angle  $\theta$  are defined as follows :—

$$\text{Sine of the angle } \theta, \text{ written as } \sin \theta = \frac{PN}{OP}$$

i.e.,  $\frac{\text{opposite side}}{\text{hypotenuse}}$

$$\text{Cosine of } \theta, \text{ written as } \cos \theta = \frac{ON}{OP}$$

i.e.,  $\frac{\text{adjacent side}}{\text{hypotenuse}}$

---

\* For alternative definitions, see *Appendix*.



Tangent of  $\theta$ , written as  $\tan \theta = \frac{PN}{ON}$

i.e.,  $\frac{\text{opposite side}}{\text{adjacent side}}$

Cosecant of  $\theta$ , written as  $\operatorname{cosec} \theta = \frac{OP}{PN}$

i.e.,  $\frac{\text{hypotenuse}}{\text{opposite side}}$

Secant of  $\theta$ , written as  $\sec \theta = \frac{OP}{ON}$

i.e.,  $\frac{\text{hypotenuse}}{\text{adjacent side}}$

Cotangent of  $\theta$ , written as  $\cot \theta = \frac{ON}{PN}$

i.e.,  $\frac{\text{adjacent side}}{\text{opposite side}}$ .

In addition to these, we define two less important ratios of the angle  $\theta$  which are sometimes used, as follows :—

Versed sine of angle  $\theta$ , written as  $\operatorname{vers} \theta = 1 - \cos \theta$ .

Coversed sine of angle  $\theta$ , written as  $\operatorname{covers} \theta = 1 - \sin \theta$ .

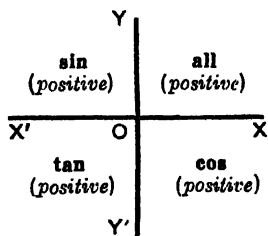
### 8. Signs of Trigonometrical ratios.

$XOP$  being any angle, traced out by a revolving line which starts from  $OX$ , it has already been mentioned in the previous Chapter that the plane may be divided into four quadrants by the two perpendicular lines  $XOX'$  and  $YOY'$ , the first being horizontal and the second vertical.

It is conventional, as in graphs, to consider distances measured along  $OX$  and  $OY$  as positive, and along  $OX'$  and  $OY'$  as negative. *The distance measured along  $OP$ , the final position of the revolving line corresponding to the angle  $XOP$ , in whichever quadrant it may lie, is however always considered positive.*

With this convention, if  $OP$  lies in the first quadrant as in Fig. (i) of the last article, the sides  $PN$ ,  $ON$  and  $OP$  of the right-angled triangle  $OPN$  are all positive. Hence, all the trigonometrical ratios are positive. If  $OP$  lies in the third quadrant as in Fig. (ii),  $ON$  and  $PN$  are both negative, but  $OP$  is positive. Hence, from the definition of the trigonometrical ratios,  $\sin XOP \left( -\frac{PN}{OP} \right)$  is negative,  $\cos XOP \left( -\frac{ON}{OP} \right)$  is negative,  $\tan XOP \left( -\frac{PN}{ON} = \frac{\text{negative quantity}}{\text{negative quantity}} \right)$  is positive etc.

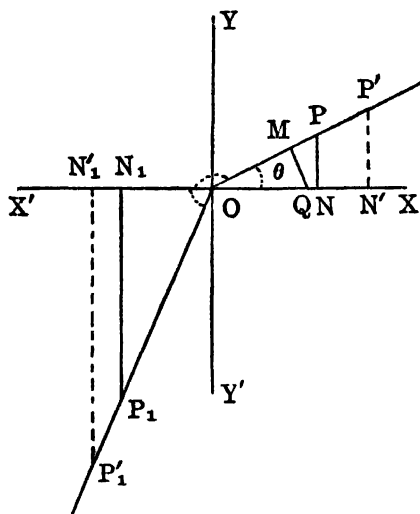
In this way, according to the final position of the revolving line (starting position being  $OX$ ), we can determine the signs of the trigonometrical ratios of the angle  $XOP$  whether this angle traced out is positive or negative. If  $OP$  is in the *first quadrant*, the ratios are all positive. If  $OP$  falls in the *second quadrant*, sine and cosecant (which is evidently the reciprocal of sine), are positive; all the other ratios are negative. If  $OP$  be in the *third quadrant*, tangent and cotangent (which are reciprocals to each other) are positive; all the others are negative. In the *fourth quadrant*, cosine and secant are positive, others are negative. A symbolical figure will help the memory in this case, namely that according to the position of  $OP$ ,



The positiveness of sine, cosine and tangent also implies the positiveness of their reciprocals namely, cosecant, secant and cotangent respectively.

### 9. Constancy of Trigonometrical ratios.

So long as an angle remains the same, its trigonometrical ratios are unique.



Let  $XOP (= \theta)$  be any angle, and let  $PN$  and  $P'N'$  be drawn perpendiculars upon  $OX$  from any two points  $P$  and  $P'$  on  $OP$ . The two right-angled triangles  $OPN$  and  $OP'N'$  are similar. Hence,  $\sin \theta$ , whether we take it as  $\frac{PN}{OP}$  or  $\frac{P'N'}{OP'}$  is the same. If the angle be  $XOP_1$ , when  $OP_1$  is not in the first quadrant, the right-angled triangles  $P_1N_1O$  and  $P'_1N'_1O$  are not only similar but also have their corresponding sides of the same sign. Hence, the trigonometrical ratios of the angle  $XOP_1$ , whether defined from the triangle  $P_1N_1O$  or from  $P'_1N'_1O$  are the same in magnitude as well as in sign. Thus for any given angle, the trigonometrical ratios are unique.

**Note.** In case of a *positive acute angle* like  $XOP$ , we might take any point  $Q$  on  $OX$  as well, and draw  $QM$  perpendicular upon  $OP$ , and define  $\sin XOP$  to be  $\frac{\text{opposite side}}{\text{hypotenuse}}$  i.e.,  $\frac{QM}{OQ}$ ,  $\cos XOP$  to be  $\frac{OM}{OQ}$  etc. Now the two triangles  $QOM$  and  $PON$  are easily seen to be similar and both have their sides all positive; so that  $\frac{QM}{OQ} = \frac{PN}{OP}$ ,  $\frac{OM}{OQ} = \frac{ON}{OP}$  etc. Hence the trigonometrical ratios of the angle  $XOP$ , even if defined from triangle  $QOM$ , will have the same values.

It may also be noted that for *angles of any magnitude, positive or negative*, any of the two arms may be supposed to be coincident with  $OX$ , and then the magnitude and sign of the angle will fix up the position of the other arm, and thereby will make the trigonometrical ratios unique.

### 10. Fundamental relations between the Trigonometrical ratios of any angle.

From the very definitions given in Art. 7 of the trigonometrical ratios of any angle  $XOP (= \theta)$  of whatever magnitude and sign, we at once derive the following relations :

$$\operatorname{cosec} \theta = \frac{1}{\sin \theta}$$

$$\sec \theta = \frac{1}{\cos \theta}$$

$$\cot \theta = \frac{1}{\tan \theta}$$

$$\text{and since } \sin \theta = \frac{PN}{OP}, \cos \theta = \frac{ON}{OP}, \tan \theta = \frac{PN}{ON}, \cot \theta = \frac{ON}{PN},$$

$$\text{we get } \tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$\cot \theta = \frac{\cos \theta}{\sin \theta}$$

Again, since in the right-angled triangle  $OPN$ ,

$$OP^2 = PN^2 + ON^2,$$

dividing by  $OP^2$ ,  $ON^2$  and  $PN^2$  respectively, we get

$$\left(\frac{PN}{OP}\right)^2 + \left(\frac{ON}{OP}\right)^2 = 1 \quad \dots \quad \dots \quad (i)$$

$$\left(\frac{OP}{ON}\right)^2 - \left(\frac{PN}{ON}\right)^2 = 1 \quad \dots \quad \dots \quad (ii)$$

$$\left(\frac{OP}{PN}\right)^2 = 1 + \left(\frac{ON}{PN}\right)^2 \quad \dots \quad \dots \quad (iii)$$

From the definition of the trigonometrical ratios,  
(i) gives

$$(\sin \theta)^2 + (\cos \theta)^2 = 1.$$

Now, it is usual to write  $(\sin \theta)^2$  in the form  $\sin^2 \theta$  and so for other ratios. The relation then reduces to the form

$$\sin^2 \theta + \cos^2 \theta = 1.$$

Similarly, (ii) and (iii) give respectively,

$$\sec^2 \theta = 1 + \tan^2 \theta$$

$$\operatorname{cosec}^2 \theta = 1 + \cot^2 \theta.$$

These formulæ are also used in the forms

$$\sin^2 \theta = 1 - \cos^2 \theta, \quad \cos^2 \theta = 1 - \sin^2 \theta,$$

$$\sec^2 \theta - \tan^2 \theta = 1, \quad \tan^2 \theta = \sec^2 \theta - 1, \text{ etc.}$$

**Note.** The fundamental formulæ derived in this article are very important, and are true for all values of  $\theta$  whatever its magnitude and sign may be. For example, if we take  $\frac{\theta}{2}$  in place of  $\theta$ , we are simply taking a different angle for which the same relations are true, so that  $\sin^2 \frac{\theta}{2} + \cos^2 \frac{\theta}{2} = 1$ , etc.

## 11. Conversions of Trigonometrical ratios.

With the help of the formulæ of the previous article, we can express any trigonometrical ratio of an angle in terms of any other ratio for the same angle ; hence if the value of any trigonometrical ratio of an angle be given, we can find the value of any other ratio.

**Ex. 1.** Express  $\sin \theta$  in terms of  $\cot \theta$ .

From the formulæ  $\operatorname{cosec} \theta = \frac{1}{\sin \theta}$

$$\text{and } \operatorname{cosec}^2 \theta = 1 + \cot^2 \theta,$$

$$\text{we get } \sin \theta = \frac{1}{\operatorname{cosec} \theta} = \pm \frac{1}{\sqrt{1 + \cot^2 \theta}}$$

**Ex. 2.** Express  $\operatorname{cosec} \theta$  in terms of  $\sec \theta$ .

$$\begin{aligned} \operatorname{cosec} \theta &= \pm \sqrt{1 + \cot^2 \theta} = \pm \sqrt{1 + \frac{1}{\tan^2 \theta}} \\ &= \pm \sqrt{\frac{\tan^2 \theta + 1}{\tan^2 \theta}} = \pm \sqrt{\frac{\sec^2 \theta}{\sec^2 \theta - 1}} = \frac{\pm \sec \theta}{\sqrt{\sec^2 \theta - 1}} \end{aligned}$$

**Ex. 3.** If  $\cos A = \frac{1}{12}$ , find  $\tan A$ .

$$\begin{aligned} \text{We have } \tan A &= \frac{\sin A}{\cos A} = \frac{\pm \sqrt{1 - \cos^2 A}}{\cos A} \\ &= \frac{\pm \sqrt{1 - \frac{1}{144}}}{\frac{1}{12}} = \frac{\pm \frac{5}{12}}{\frac{1}{12}} = \pm \frac{5}{12} \end{aligned}$$

A more practical method in such cases is however to construct a right-angled triangle with the numerator and denominator as the two suitable sides, as shown below.

**Ex. 4.** If  $\sec A = \frac{41}{9}$ , find  $\cot A$ ,

Let  $APN$  be a triangle right-angled at  $N$  in which the hypotenuse  $AP = 41$ ,  $AN = 9$ ,

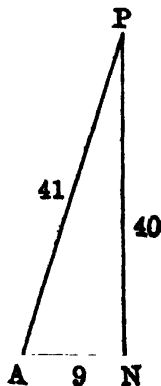
$$\text{so that } \sec NAP = \frac{AP}{AN} = \frac{41}{9}.$$

Thus  $\angle NAP = A$ .

$$\begin{aligned} \text{Now, } PN^2 &= AP^2 - AN^2 = 41^2 - 9^2 \\ &= 40^2, \end{aligned}$$

so that  $PN = \pm 40$ .

$$\therefore \cot A = \cot NAP = \frac{AN}{PN} = \pm \frac{9}{40}.$$



### 12. Restrictions on the magnitudes of Trigonometrical ratios.

From the relation  $\sin^2 \theta + \cos^2 \theta = 1$ , since  $\sin^2 \theta$  and  $\cos^2 \theta$  being square quantities are both positive, it is evident that neither  $\sin^2 \theta$  nor  $\cos^2 \theta$  can exceed 1, for if  $\sin^2 \theta$ , for example, be greater than 1,  $\cos^2 \theta$  (which is a square quantity) becomes negative, which is impossible. Thus,  $\sin \theta$  as well as  $\cos \theta$  must have numerical values not exceeding 1; in other words, both  $\sin \theta$  and  $\cos \theta$  must lie between +1 and -1 whatever the magnitude of  $\theta$  may be. Any value numerically greater than 1, like -2 or +3.1 must be impossible for  $\sin \theta$  or  $\cos \theta$  so long  $\theta$  is real.

$\sec \theta$  and  $\operatorname{cosec} \theta$  therefore, being reciprocals of  $\cos \theta$  and  $\sin \theta$  respectively, can never be numerically less than 1.

$\tan \theta$  and  $\cot \theta$  however, can have any numerical value greater than 1 or less than 1 according to the value of  $\theta$ .

13. A few examples on the applications of the fundamental formulæ are given below.

Ex. 1. Prove that  $\sqrt{\frac{1+\cos \theta}{1-\cos \theta}} = \operatorname{cosec} \theta + \cot \theta$ .

[C. U. 1937]

$$\begin{aligned}\sqrt{\frac{1+\cos \theta}{1-\cos \theta}} &= \sqrt{\frac{(1+\cos \theta)^2}{1-\cos^2 \theta}} = \sqrt{\frac{(1+\cos \theta)^2}{\sin^2 \theta}} \\ &= \frac{1+\cos \theta}{\sin \theta} = \frac{1}{\sin \theta} + \frac{\cos \theta}{\sin \theta} = \operatorname{cosec} \theta + \cot \theta.\end{aligned}$$

Ex. 2. Prove that

$$\frac{1}{\sec A + \tan A} - \frac{1}{\cos A} = \frac{1}{\cos A} - \frac{1}{\sec A - \tan A}.$$

$$\begin{aligned}\text{We have } & \frac{1}{\sec A + \tan A} + \frac{1}{\sec A - \tan A} \\ &= \frac{\sec A - \tan A + \sec A + \tan A}{(\sec A + \tan A)(\sec A - \tan A)} = \frac{2 \sec A}{\sec^2 A - \tan^2 A} \\ &= 2 \sec A = \frac{2}{\cos A} = \frac{1}{\cos A} + \frac{1}{\cos A}.\end{aligned}$$

Hence, by transposition,

$$\frac{1}{\sec A + \tan A} - \frac{1}{\cos A} = \frac{1}{\cos A} - \frac{1}{\sec A - \tan A}.$$

**Ex. 3.** Prove that  $\frac{1 + 2 \sin \theta \cos \theta}{(\sin \theta + \cos \theta)(\cot \theta + \tan \theta)}$   
 $= \sin \theta \cos \theta (\sin \theta + \cos \theta).$

$$\begin{aligned} \text{We have } & \frac{1 + 2 \sin \theta \cos \theta}{(\sin \theta + \cos \theta)(\cot \theta + \tan \theta)} \\ &= \frac{(\sin^2 \theta + \cos^2 \theta) + 2 \sin \theta \cos \theta}{(\sin \theta + \cos \theta) \left( \frac{\cos \theta}{\sin \theta} + \frac{\sin \theta}{\cos \theta} \right)} \\ &= \frac{(\sin \theta + \cos \theta)^2}{(\sin \theta + \cos \theta) \left( \frac{\cos^2 \theta + \sin^2 \theta}{\sin \theta \cos \theta} \right)} \\ &= \frac{(\sin \theta + \cos \theta) \sin \theta \cos \theta}{1} \\ &= \sin \theta \cos \theta (\sin \theta + \cos \theta). \end{aligned}$$

**Ex. 4.** If  $15 \sin^2 \theta + 2 \cos \theta = 7$ , find  $\tan \theta$ .

Here  $15(1 - \cos^2 \theta) + 2 \cos \theta = 7$ ,

whence  $15 \cos^2 \theta - 2 \cos \theta - 8 = 0$ ,

or,  $(5 \cos \theta - 4)(3 \cos \theta + 2) = 0$ ,  $\therefore \cos \theta = \frac{4}{5}$ , or,  $-\frac{2}{3}$ .

Case (i) when  $\cos \theta = \frac{4}{5}$ ,

$$\sin^2 \theta = 1 - \cos^2 \theta = 1 - \frac{16}{25} = \frac{9}{25}. \quad \therefore \sin \theta = \pm \frac{3}{5},$$

and so  $\tan \theta = \frac{\sin \theta}{\cos \theta} = \pm \frac{3}{4}$ .

Case (ii) when  $\cos \theta = -\frac{2}{3}$ ,

$$\sin^2 \theta = 1 - \cos^2 \theta = 1 - \frac{4}{9} = \frac{5}{9}. \quad \therefore \sin \theta = \pm \frac{\sqrt{5}}{3}.$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta} = \pm \frac{\sqrt{5}}{2}$$



## Examples II

Prove the following identities (*Ex. 1 to 24*) :-

1.  $\frac{\sin A + \cos A}{\sec A + \operatorname{cosec} A} = \sin A \cos A.$
2.  $\cot \theta + \tan \theta = \sec \theta \operatorname{cosec} \theta.$
3.  $\frac{1}{1 + \tan A} = \frac{\cot A}{1 + \cot A}.$
4.  $\operatorname{cosec}^6 A - \cot^6 A = 1 + 3 \operatorname{cosec}^2 A \cot^2 A.$
5.  $\cos^6 A + \sin^6 A = 1 - 3 \sin^2 A \cos^2 A.$
6.  $\frac{1}{\cos^2 A} - \frac{1}{\operatorname{cosec}^2 A - 1} = 1.$
7.  $\cos A + \tan A \sin A = \sec A.$
8.  $\sec^4 A + \tan^4 A = 1 + 2 \sec^2 A \tan^2 A.$
9.  $\frac{1 + 3 \cos \theta - 4 \cos^3 \theta}{1 - \cos \theta} = (1 + 2 \cos \theta)^2.$
10.  $(\cot \theta + \operatorname{cosec} \theta)^2 = \frac{1 + \cos \theta}{1 - \cos \theta}.$
11.  $\frac{1 + \tan^2 \theta}{1 + \cot^2 \theta} = \left( \frac{1 - \tan \theta}{1 - \cot \theta} \right)^2.$
12.  $\frac{\tan^2 \alpha - \cot^2 \alpha}{1 + \cot^2 \alpha} = \frac{\sin^2 \alpha - \cos^2 \alpha}{\cos^2 \alpha}.$
13.  $1 + \tan \theta + \sec \theta = \frac{2}{1 + \cot \theta - \operatorname{cosec} \theta}.$
14.  $\frac{\tan \theta + \sec \theta - 1}{\tan \theta - \sec \theta + 1} = \frac{1 + \sin \theta}{\cos \theta}.$
15.  $\frac{\sin A - 2 \sin^3 A}{2 \cos^3 A - \cos A} = \tan A.$
16.  $\sqrt{\frac{1 + \sin \theta}{1 - \sin \theta}} - \sec \theta = \sec \theta - \sqrt{\frac{1 - \sin \theta}{1 + \sin \theta}}.$

$$17. \frac{\operatorname{cosec} A + \cot A}{\operatorname{cosec} A - \cot A} = \frac{\sin^2 A}{(1 - \cos A)^2}.$$

$$18. (1 + \sin A + \cos A)^2 = 2(1 + \sin A)(1 + \cos A).$$

$$19. \frac{\sec \theta + \tan \theta}{\operatorname{cosec} \theta + \cot \theta} - \frac{\sec \theta - \tan \theta}{\operatorname{cosec} \theta - \cot \theta} = 2(\sec \theta - \operatorname{cosec} \theta).$$

$$20. \frac{1}{1 + \sin^2 \theta} + \frac{1}{1 + \operatorname{cosec}^2 \theta} = 1.$$

$$21. \frac{\sin^3 \alpha + \cos^3 \alpha}{\sin \alpha + \cos \alpha} + \frac{\sin^3 \alpha - \cos^3 \alpha}{\sin \alpha - \cos \alpha} = 2.$$

$$22. \frac{\tan \theta}{\sec \theta - 1} - \frac{\sin \theta}{1 + \cos \theta} = 2 \cot \theta.$$

$$23. \frac{\cos \theta + \cos \phi}{\sin \theta - \sin \phi} = \frac{\sin \theta + \sin \phi}{\cos \phi - \cos \theta}.$$

$$24. 1 + 4 \operatorname{cosec}^2 \theta \cot^2 \theta = (\operatorname{cosec}^2 \theta + \cot^2 \theta)^2.$$

$$25. \text{Express } 1 - 2 \sin \theta \cos \theta \text{ as a perfect square.}$$

$$26. \text{Express } 2 \sec^2 \theta - \sec^4 \theta - 2 \operatorname{cosec}^2 \theta + \operatorname{cosec}^4 \theta \text{ in terms of } \tan \theta.$$

$$27. \text{Prove that}$$

$$(\sin \alpha \cos \beta + \cos \alpha \sin \beta)(\sin \alpha \cos \beta - \cos \alpha \sin \beta) \\ = \sin^2 \alpha - \sin^2 \beta.$$

$$28. \text{If } \sin A + \sin^2 A = 1, \text{ then } \cos^2 A + \cos^4 A = 1.$$

$$29. (i) \text{ If } \sin \theta - \cos \theta = 0, \text{ prove that } \sec \theta = \pm \sqrt{2}.$$

$$(ii) \text{ If } 7 \sin^2 \theta + 3 \cos^2 \theta = 4, \text{ show that } \tan \theta = \pm \frac{1}{\sqrt{3}}.$$

$$(iii) \text{ If } 3 \sin \theta + 4 \cos \theta = 5, \text{ show that } \sin \theta = \frac{3}{5}.$$

$$30. \text{ If } \tan \theta + \sec \theta = x, \text{ show that } \sin \theta = \frac{x^2 - 1}{x^2 + 1}.$$

$$31. \text{ If } \tan \theta = \frac{a}{b}, \text{ find the value of } \frac{a \sin \theta - b \cos \theta}{a \sin \theta + b \cos \theta}.$$

32. If  $1 + 4x^3 = 4x \sec A$ , prove that  
 $\sec A + \tan A = 2x$  or  $1/2x$ .
33. Express  $\sin \alpha$  in terms of  $\sec \alpha$ , and  $\sec \theta$  in terms of  $\cot \theta$ .
34. Given  $\sin \theta = \frac{3}{5}$ ,  $\cos \phi = \frac{1}{\sqrt{2}}$ , where  $\theta$  and  $\phi$  are acute angles, find the value of  $\frac{\tan \theta - \tan \phi}{1 + \tan \theta \tan \phi}$ .
35. If  $\cos \alpha + \sin \alpha = \sqrt{2} \cos \alpha$ , prove that  
 $\cos \alpha - \sin \alpha = \sqrt{2} \sin \alpha$ .
36. If  $\tan A = \frac{1}{\sqrt{3}}$ , find  $\frac{\operatorname{cosec}^2 A - \sec^2 A}{\operatorname{cosec}^2 A + \sec^2 A}$ .
37. If  $1 + \sin^2 A = 3 \sin A \cos A$ , find  $\tan A$ .
38. If  $\tan \theta + \sin \theta = m$ ,  $\tan \theta - \sin \theta = n$ , prove that  
 $m^2 - n^2 = 4 \sqrt{mn}$ .
39. If  $(a^2 - b^2) \sin \theta + 2ab \cos \theta = a^2 + b^2$ , find  $\tan \theta$  and  $\operatorname{cosec} \theta$ .
40. If  $\tan \theta = \frac{\sin \alpha - \cos \alpha}{\sin \alpha + \cos \alpha}$ , prove that  
 $\sqrt{2} \cos \theta = \sin \alpha + \cos \alpha$ .
41. If  $\cos^4 \theta + \cos^2 \theta = 1$ , show that  
 (i)  $\tan^4 \theta + \tan^2 \theta = 1$ .  
 (ii)  $\cot^4 \theta - \cot^2 \theta = 1$ .
42. If  $x$  and  $y$  are two unequal real quantities, show that the equations (i)  $\sin^2 \theta = \frac{(x+y)^2}{4xy}$  and (ii)  $\cos \theta = x + \frac{1}{x}$  are both impossible.
43. Eliminate  $\theta$  between  
 (i)  $x = a \cos \theta$ ,  $y = b \sin \theta$ .  
 (ii)  $x = c (\sec \theta + \tan \theta)$ ,  $y = c (\sec \theta - \tan \theta)$ .  
 (iii)  $a \cos \theta + b \sin \theta + c = 0$ ,  $a' \cos \theta + b' \sin \theta + c' = 0$ .  
 (iv)  $a \tan^2 \theta + b \tan \theta + c = a' \cot^2 \theta + b' \cot \theta + c' = 0$ .

## Examples II(A)

Prove the following identities (*Ex. 1 to 18*) :—

1.  $\frac{\tan^3 \alpha}{1 + \tan^2 \alpha} + \frac{\cot^3 \alpha}{1 + \cot^2 \alpha} = \frac{1 - 2 \sin^2 \alpha \cos^2 \alpha}{\sin \alpha \cos \alpha}.$
2.  $(\tan \theta + \cot \theta + \sec \theta)(\tan \theta + \cot \theta - \sec \theta) = \operatorname{cosec}^2 \theta.$
3.  $\sin \theta (1 + \tan \theta) + \cos \theta (1 + \cot \theta) = \sec \theta + \operatorname{cosec} \theta.$   
[ C. U 1935 ]
4.  $(1 + \sin \alpha - \cos \alpha)^2 + (1 - \sin \alpha + \cos \alpha)^2$   
 $= 4(1 - \sin \alpha \cos \alpha).$
5.  $\sin^6 \alpha + \sin^4 \alpha \cos^2 \alpha - \sin^2 \alpha \cos^4 \alpha - \cos^6 \alpha$   
 $= \sin^2 \alpha - \cos^2 \alpha.$
6.  $3(\sin \theta + \cos \theta) - 2(\sin^3 \theta + \cos^3 \theta) = (\sin \theta + \cos \theta)^3.$
7.  $\frac{1}{\operatorname{cosec} \theta - \cot \theta} - \frac{1}{\sin \theta} = \frac{1}{\sin \theta} - \frac{1}{\operatorname{cosec} \theta + \cot \theta}.$
8.  $\frac{\cos x}{\sin x + \cos y} + \frac{\cos y}{\sin y - \cos x} = \frac{\cos x}{\sin x - \cos y} + \frac{\cos y}{\sin y + \cos x}.$
9.  $(\sin \theta + \operatorname{cosec} \theta)^2 + (\cos \theta + \sec \theta)^2 = \tan^2 \theta + \cot^2 \theta + 7.$
10.  $(\sec \theta - \cos \theta)(\operatorname{cosec} \theta - \sin \theta)(\tan \theta + \cot \theta) = 1.$
11.  $\frac{1 + (\operatorname{cosec} x \tan y)^2}{1 + (\operatorname{cosec} z \tan y)^2} = \frac{1 + (\cot x \sin y)^2}{1 + (\cot z \sin y)^2}.$
12.  $\sec^3 \alpha \operatorname{cosec}^3 \alpha - 3 \sec \alpha \operatorname{cosec} \alpha = \tan^3 \alpha + \cot^3 \alpha.$
13.  $\sin^6 A - \cos^6 A = (\sin A + \cos A)(\sin A - \cos A)$   
 $\times (1 + \sin A \cos A)(1 - \sin A \cos A).$
14.  $\frac{\tan \alpha}{(1 + \tan^2 \alpha)^2} + \frac{\cot \alpha}{(1 + \cot^2 \alpha)^2} = \sin \alpha \cos \alpha.$
15.  $\sin^2 \theta \tan \theta - \cos^2 \theta \cot \theta + \sec \theta \operatorname{cosec} \theta = 2 \tan \theta.$
16.  $\frac{\cos^3 A - \sin^2 A}{\sin A \cos^2 A - \cos A \sin^2 A} = \operatorname{cosec} A + \sec A.$
17.  $\frac{\tan^3 A + \cot^2 A}{\tan^2 A - \cot^2 A} = \frac{\sin^4 A + \cos^4 A}{\sin^2 A - \cos^2 A}.$

$$18. (\sin \alpha \cos \beta - \cos \alpha \sin \beta)^2 \\ + (\cos \alpha \cos \beta + \sin \alpha \sin \beta)^2 = 1.$$

$$19. \text{ If } \cos^2 A - \sin^2 A = \tan^2 B, \\ \text{ then } \cos^2 B - \sin^2 B = \tan^2 A.$$

$$20. \text{ If } \sin^4 x + \sin^2 x = 1, \text{ the } \tan^4 x - \tan^2 x = 1.$$

21. Show that the difference between  $3 \sin^4 \theta - 2 \sin^6 \theta$  and  $2 \cos^6 \theta - 3 \cos^4 \theta$  is the same for all values of  $\theta$ .

$$22. \text{ If } x = \frac{1 + \sin \theta}{\cos \theta}, \text{ show that } \frac{1}{x} = \frac{1 - \sin \theta}{\cos \theta}.$$

$$23. \text{ If } \tan^2 A = 1 + 2 \tan^2 B, \text{ show that } \cos^2 B = 2 \cos^2 A.$$

$$24. \text{ If } \sin \alpha + \cos \alpha = 1, \text{ the } \sin \alpha - \cos \alpha = \pm 1.$$

$$25. \text{ If } a \cos \theta - b \sin \theta = c, \text{ then show that} \\ a \sin \theta + b \cos \theta = \pm \sqrt{a^2 + b^2 - c^2}.$$

$$26. \text{ If } (1 + \sin x)(1 + \sin y)(1 + \sin z) \\ = (1 - \sin x)(1 - \sin y)(1 - \sin z),$$

prove that each is equal to  $\pm \cos x \cos y \cos z$ .

$$27. \text{ If } x \sin^3 \alpha + y \cos^3 \alpha = \sin \alpha \cos \alpha, \text{ and} \\ x \sin \alpha - y \cos \alpha = 0, \text{ then } x^2 + y^2 = 1. \quad [ C. U. 1937 ]$$

$$28. \text{ If } \sin A = \frac{\sin x + \sin y}{1 + \sin x \sin y}, \text{ show that} \\ \cos A = \pm \frac{\cos x \cos y}{1 + \sin x \sin y}.$$

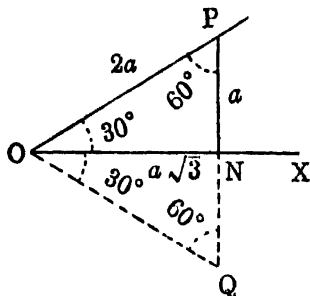
$$29. (i) \text{ If } \sin \alpha + \operatorname{cosec} \alpha = 2, \\ \text{ then } \sin^n \alpha + \operatorname{cosec}^n \alpha = 2.$$

$$(ii) \text{ If } \sec \alpha = \sec \beta \sec \gamma + \tan \beta \tan \gamma, \\ \text{ then } \sec \beta = \sec \gamma \sec \alpha \pm \tan \gamma \tan \alpha.$$

$$30. \text{ If } \frac{\cos^4 x}{\cos^2 y} + \frac{\sin^4 x}{\sin^2 y} = 1, \text{ then } \frac{\cos^4 y}{\cos^2 x} + \frac{\sin^4 y}{\sin^2 x} = 1.$$

CHAPTER III  
TRIGONOMETRICAL RATIOS OF SOME  
STANDARD ANGLES

**14. Ratios of  $30^\circ$ .**



Let the angle  $XOP$ , which may be supposed to be traced out by a revolving line starting from  $OX$ , be  $30^\circ$ . Let  $PN$  be drawn perpendicular upon  $OX$  from any point  $P$  on  $OP$ . The angle  $OPN$  is then  $60^\circ$ .

Produce  $PN$  to  $Q$ , making  $NQ = NP$ . Join  $OQ$ . The triangles  $PON$  and  $QON$  are easily seen to be equal in all respects, and so  $\angle OQN = \angle OPN = 60^\circ$ . Hence, the triangle  $OPQ$  is equilateral, and so  $OP = PQ =$  double of  $PN$ .

Hence, in the above figure, if  $PN = a$ , then  $OP = 2a$  and so  $ON = \sqrt{OP^2 - PN^2} = \sqrt{4a^2 - a^2} = \sqrt{3}a$ . The sides  $ON$ ,  $PN$  and  $OP$  are all positive in this case, since the angle is acute.

Hence,

$$\sin 30^\circ = \sin PON = \frac{PN}{OP} = \frac{a}{2a} = \frac{1}{2}$$

$$\cos 30^\circ = \frac{ON}{OP} = \frac{\sqrt{3}a}{2a} = \frac{\sqrt{3}}{2}$$

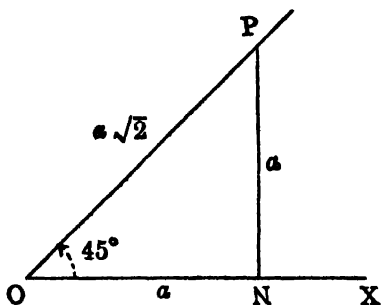
$$\tan 30^\circ = \frac{PN}{ON} = \frac{1}{\sqrt{3}}$$

$$\cot 30^\circ = \frac{ON}{PN} = \sqrt{3}$$

$$\operatorname{cosec} 30^\circ = \frac{1}{\sin 30^\circ} = 2$$

$$\sec 30^\circ = \frac{1}{\cos 30^\circ} = \frac{2}{\sqrt{3}}$$

### 15. Ratios of $45^\circ$ .



Let  $\angle XOP = 45^\circ$ .  $PN$  is perpendicular on  $OX$ . In the right-angled triangle  $PON$ ,  $\angle PON = 45^\circ$ .

Therefore,  $\angle OPN$  is also  $45^\circ$  and so  $ON = PN = a$  suppose.  
Then  $OP = \sqrt{ON^2 + PN^2} = \sqrt{a^2 + a^2} = a\sqrt{2}$ .

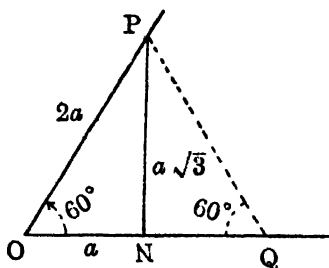
Hence,

$$\sin 45^\circ = \frac{PN}{OP} = \frac{1}{\sqrt{2}}$$

$$\cos 45^\circ = \frac{ON}{OP} = \frac{1}{\sqrt{2}}$$

$$\tan 45^\circ = \frac{PN}{ON} = 1$$

$$\sec 45^\circ = \operatorname{cosec} 45^\circ = \sqrt{2}, \cot 45^\circ = 1.$$

16. Ratios of  $60^\circ$ .

Let  $\angle XOP = 60^\circ$ . Now  $PN$  being perpendicular upon  $OX$ , along  $NX$  out off  $NQ = ON$ . Join  $PQ$ . Then the two triangles  $OPN$  and  $QPN$  are easily seen to be congruent. Hence,  $\angle PQN = \angle PON = 60^\circ$ . Thus, the triangle  $POQ$  is equilateral, and so  $OP = OQ = \text{double of } ON$ .

If  $ON = a$ , then  $OP = 2a$  and hence  $PN = \sqrt{OP^2 - ON^2} = a\sqrt{3}$ .

$$\text{Then} \quad \sin 60^\circ = \frac{PN}{OP} = \frac{\sqrt{3}}{2}$$

$$\cos 60^\circ = \frac{ON}{OP} = \frac{1}{2}$$

$$\tan 60^\circ = \frac{PN}{ON} = \sqrt{3}$$

$$\cot 60^\circ = \frac{1}{\sqrt{3}}, \sec 60^\circ = 2, \operatorname{cosec} 60^\circ = \frac{2}{\sqrt{3}}.$$

**Note.** It may be noted from the values of the ratios that  $\sin 60^\circ = \cos 30^\circ$ ,  $\cos 60^\circ = \sin 30^\circ$ ,  $\tan 60^\circ = \cot 30^\circ$ ,  $\cot 60^\circ = \tan 30^\circ$ ,  $\sec 60^\circ = \operatorname{cosec} 30^\circ$ ,  $\operatorname{cosec} 60^\circ = \sec 30^\circ$ . It will be proved more generally, in the next chapter, that for any two complementary angles sine of one is the cosine of the other and *vice versa*, tangent of one is the cotangent of the other, and secant of one is the cosecant of the other. The angle  $45^\circ$  being its own complement, therefore, it should have its sine and cosine equal to one another, as is actually seen to be the case.



17. Ratios of  $90^\circ$ .

Let  $XOP$  be an acute angle very nearly  $90^\circ$ .  $PN$  being

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perpendicular upon  $OX$ ,  $ON$  is extremely small, and as  $\angle XOP$  approaches more and more to  $90^\circ$ ,  $ON$  becomes smaller and smaller. The length  $OP$  may however remain finite, and  $PN$  and  $OP$  will approach each other more and more closely. Ultimately when  $\angle XOP$  becomes  $90^\circ$ ,  $OP$  and  $PN$  coincide, and  $ON$  becomes zero ultimately. Hence the ratio

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$PN/OP$  becomes 1 and  $ON/OP$  becomes zero.

Thus,  $\sin 90^\circ = \frac{PN}{OP}$  in the limit = 1

$\cos 90^\circ = \frac{ON}{OP}$  in the limit = 0

$\tan 90^\circ = \frac{PN}{ON}$  in the limit =  $\infty^*$  (infinity)

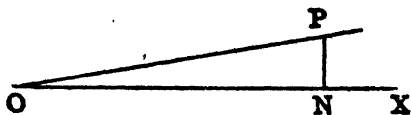
( since  $ON \rightarrow 0$ , whereas  $PN$  remains finite )

$\cot 90^\circ = \frac{\cos 90^\circ}{\sin 90^\circ} = \frac{0}{1} = 0$

$\operatorname{cosec} 90^\circ = 1, \sec 90^\circ = OP/ON$  in the limit =

\*The symbol  $\infty$  is used to denote a quantity which exceeds any positive number, however large, and does not represent a definite number.

It should be noted that in determining  $\tan 90^\circ$ , we may start with an angle  $XOP$ , slightly greater than  $90^\circ$ , (i.e., in the second quadrant), and make it approach  $90^\circ$ . Then  $ON$  will be negative and  $\rightarrow 0$ , whereas  $PN$  is positive. Accordingly we may also write  $\tan 90^\circ = -\infty$ . Thus strictly speaking, we should write  $\tan 90^\circ = \pm \infty$ . Similar remarks apply for  $\sec 90^\circ$ ,  $\cot 0^\circ$ ,  $\operatorname{cosec} 0^\circ$ .

18. Ratios of  $0^\circ$ .

Let  $\angle XOP$  be an infinitely small positive angle, and let  $PN$  be perpendicular on  $OX$ .

Then,  $PN$  is infinitely small, whereas  $OP$  is finite. Now, if  $\angle XOP$  be taken less and less and ultimately becomes less than any quantity we can assign, we denote it by zero, and in this case  $PN$  practically vanishes, whereas  $OP$  and  $ON$  remaining finite, coincide. Hence, the ratio  $PN/OP$  becomes ultimately zero, and  $ON/OP$  becomes 1.

$$\text{Hence, } \sin 0^\circ = \frac{PN}{OP} \text{ in the limit} = 0$$

$$\cos 0^\circ = \frac{ON}{OP} \text{ in the limit} = 1$$

$$\tan 0^\circ = \frac{\sin 0^\circ}{\cos 0^\circ} = \frac{0}{1} = 0$$

$$\cot 0^\circ = \frac{ON}{PN} \text{ in the limit} = \infty^*$$

$$\operatorname{cosec} 0^\circ = \frac{OP}{PN} \text{ in the limit} = \infty^*$$

$$\sec 0^\circ = \frac{1}{\cos 0^\circ} = \frac{1}{1} = 1.$$

**Note.** Note that  $0^\circ$  and  $90^\circ$  being complementary,  
 $\sin 0^\circ = \cos 90^\circ$ ,  $\cos 0^\circ = \sin 90^\circ = 1$ , etc.

19. As the ratios of the standard angles  $0^\circ$ ,  $30^\circ$ ,  $45^\circ$ ,  $60^\circ$ , and  $90^\circ$  are very often used, they should be remembered very

---

\* See foot note of Art. 17.

carefully. The first three ratios are given in the tabulated form below. The other three are reciprocals to these.

angle	sine	cosine	tangent
$0^\circ$ or $0^\circ$	0	1	0
$30^\circ$ or $\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$45^\circ$ or $\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$60^\circ$ or $\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$90^\circ$ or $\frac{\pi}{2}$	1	0	$\pm \infty$

Note. The following device may be of use in remembering the *sines* and *cosines* of standard angles. The sines of the angles  $0^\circ$ ,  $30^\circ$ ,  $45^\circ$ ,  $60^\circ$ ,  $90^\circ$  are respectively the square roots of the fractions

$$0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$$

and cosines of these angles are the square roots from right to left.

## 20. Examples worked out.

Ex. 1. If  $\theta = 30^\circ$ , verify that  $\cos 2\theta = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta}$ .

Here,  $\cos 2\theta = \cos 60^\circ = \frac{1}{2}$ . Also  $\tan \theta = \tan 30^\circ = \frac{1}{\sqrt{3}}$ ,

$$\therefore \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} = \frac{1 - \frac{1}{3}}{1 + \frac{1}{3}} = \frac{1}{2}.$$

Hence,  $\cos 2\theta = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta}$ .

**Ex. 2.** *Verify that*

$$\sin 30^\circ = \sin 60^\circ \cos 30^\circ - \cos 60^\circ \sin 30^\circ.$$

The right-hand side, on substitution of the values,

$$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4} - \frac{1}{4} = \frac{2}{4} = \sin 30^\circ.$$

Hence the result.

**Ex. 3.** *Solve for  $\theta$ , where  $\theta$  is a positive acute angle, given  $\operatorname{cosec} \theta \cot \theta = 2\sqrt{3}$ .*

$$\text{From the given equation, } \frac{1}{\sin \theta} \cdot \frac{\cos \theta}{\sin \theta} = 2\sqrt{3},$$

$$\text{or, } \cos \theta = 2\sqrt{3} \sin^2 \theta = 2\sqrt{3} (1 - \cos^2 \theta),$$

$$\text{whence, } 2\sqrt{3} \cos^2 \theta + \cos \theta - 2\sqrt{3} = 0$$

$$\text{giving } \cos \theta = \frac{-1 \pm \sqrt{1+48}}{4\sqrt{3}} = \frac{-1 \pm 7}{4\sqrt{3}}.$$

Since  $\theta$  is a positive acute angle,  $\cos \theta$  is positive, and so rejecting the negative value,

$$\cos \theta = \frac{6}{4\sqrt{3}} = \frac{\sqrt{3}}{2} = \cos 30^\circ. \therefore \theta = 30^\circ \text{ i.e., } \frac{\pi}{6}.$$

### Examples III

Verify the results (Ex. 1 to 6) :

$$1. \quad 1 - 2 \sin^2 30^\circ = 2 \cos^2 30^\circ - 1 = \cos 60^\circ.$$

$$2. \quad \frac{2 \tan 30^\circ}{1 - \tan^2 30^\circ} = \sqrt{3}.$$

$$3. \quad \cos \frac{\pi}{3} \cos \frac{\pi}{4} + \sin \frac{\pi}{3} \sin \frac{\pi}{4} = \sin \frac{\pi}{4} \cos \frac{\pi}{6} + \cos \frac{\pi}{4} \sin \frac{\pi}{6}.$$

$$4. \quad (i) \quad \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

$$(ii) \quad \cos A = \cos^2 B - \sin^2 B, \text{ where } A = 60^\circ, B = 30^\circ.$$

5.  $\sin 3A = 3 \sin A - 4 \sin^3 A$ , where  $A = \frac{\pi}{6}$ .

6.  $\operatorname{cosec}^2 45^\circ \cdot \sec^2 30^\circ (\sin^2 30^\circ + 4 \cot^2 45^\circ - \sec^2 60^\circ) = \frac{1}{3}$ .

7. If  $\tan^2 \frac{\pi}{4} - \cos^2 \frac{\pi}{3} = x \sin \frac{\pi}{4} \cos \frac{\pi}{4} \tan \frac{\pi}{3}$ , find  $x$ .

8. If  $\theta$  be a positive acute angle, find  $\theta$ , when

(i)  $2 \sin^2 \theta = 3 \cos \theta$ .

(ii)  $\tan \theta + \cot \theta = 2$ .

(iii)  $\operatorname{cosec}^2 \theta + 5 = 3 \sqrt{3} \cot \theta$ .

(iv)  $\sin \theta + \cos \theta = \sqrt{2}$ .

(v)  $2(\cos^2 \theta - \sin^2 \theta) = 1$ .

(vi)  $6 \sin^2 \theta - 11 \sin \theta + 4 = 0$ .

(vii)  $\frac{\sin \theta - \cos \theta}{\sin \theta + \cos \theta} = \frac{1 - \sqrt{3}}{1 + \sqrt{3}}$ .

9. Given  $\theta$  and  $\phi$  to be positive acute angles, and  $\tan(\theta + \phi) = \sqrt{3}$ ,  $\tan(\theta - \phi) = 1$ , determine  $\theta$  and  $\phi$ .

10. Find  $\alpha$  and  $\beta$  ( $\alpha$  and  $\beta$  being positive acute angles), if

$$\sin(2\alpha - \beta) = 1,$$

$$\text{and } \cos(\alpha + \beta) = \frac{1}{2}.$$

11. Find  $A, B, C$  ( $A, B, C$  being positive acute angles), if

$$\sin(B + C - A) = 1,$$

$$\cos(C + A - B) = 1,$$

$$\text{and } \tan(A + B - C) = 1.$$

12. Find the numerical values of :—

(i)  $\cot^2 \frac{\pi}{6} - 2 \cos^2 \frac{\pi}{3} - \frac{3}{4} \sec^2 \frac{\pi}{4} - 4 \sec^2 \frac{\pi}{6}$ .

(ii)  $3 \tan^2 45^\circ - \sin^2 60^\circ - \frac{1}{2} \cot^2 30^\circ + \frac{1}{2} \sec^2 45^\circ$ .

# CHAPTER IV

## TRIGONOMETRICAL RATIOS OF ANGLES ASSOCIATED WITH A GIVEN ANGLE $\theta$

21. Ratios of the angle  $(-\theta)$  in terms of those of  $\theta$ ,  $\theta$  having any magnitude.

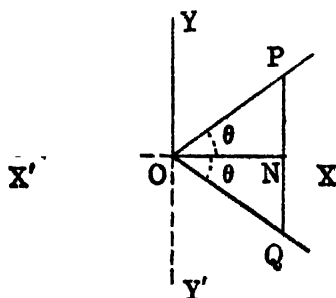


Fig. (i)

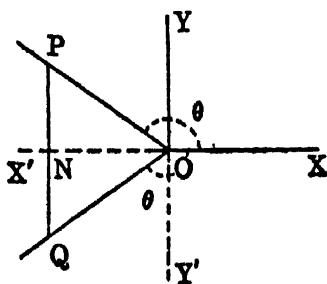


Fig. (ii)

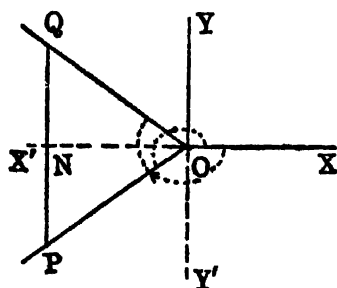


Fig. (iii)

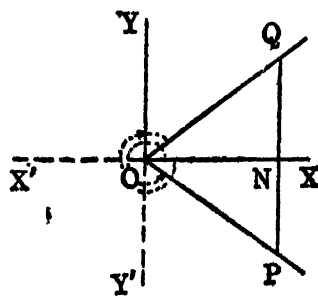


Fig. (iv)

Let the  $\angle XOP$  be  $\theta$  and the  $\angle XOQ$  described clockwise be  $-\theta$ . From any point  $P$  on  $OP$  draw  $PN$  perpendicular to  $OX$  [or  $OX'$  as in Figs. (ii) and (iii)], and produce it to meet  $OQ$  at  $Q$  say.

Now,  $\angle XOP$  (measured anti-clockwisely) being equal to  $\angle XOQ$  (measured clockwisely),  $\angle PON = \angle QON$  in magnitude in all the figures, and therefore, the two rt.-angled triangles  $PON$  and  $QON$  are congruent. The corresponding sides are therefore equal in magnitude. Considering the signs of these sides according to the usual convention, we get in all the figures,

$$QN = -PN, \text{ and } OQ = OP.$$

( both  $OP$  and  $OQ$  being always considered positive )

Hence, from definition,

$$\sin(-\theta) = \frac{QN}{OQ} = -\frac{PN}{OP} = -\sin \theta$$

$$\cos(-\theta) = \frac{ON}{OQ} = \frac{ON}{OP} = \cos \theta$$

$$\tan(-\theta) = \frac{QN}{ON} = -\frac{PN}{ON} = -\tan \theta$$

and the reciprocals of these give,

$$\operatorname{cosec}(-\theta) = -\operatorname{cosec} \theta,$$

$$\sec(-\theta) = \sec \theta,$$

$$\cot(-\theta) = -\cot \theta.$$

## 22. Ratios of $(90^\circ - \theta)$ .

Let the  $\angle XOP$  traced out by a revolving line be  $\theta$ , and let another revolving line, starting from  $OX$  trace out the angle  $XOY = 90^\circ$  and then revolve back, tracing out  $\angle YOQ = \theta$  in the clockwise direction, so that  $\angle XOQ = 90^\circ - \theta$ .

Take two equal lengths  $OP$  and  $OQ$  along  $OP$  and  $OQ$  respectively and draw  $PN$  and  $QM$  perpendiculars on  $OX$ .

If  $OP$  be in the first or third quadrant as in Fig. (i) and Fig. (iii),  $OQ$  also lies in the same quadrant. If  $OP$  lies in the second quadrant as in Fig. (ii),  $OQ$  lies in the fourth quadrant; and if  $OP$  lies in the fourth,  $OQ$  lies in the

second, as in Fig. (iv). Now,  $\angle XOP$  being equal to  $\angle YOQ$  in magnitude,  $\angle PON = \angle OQM$ , and since  $OP = OQ$ ,

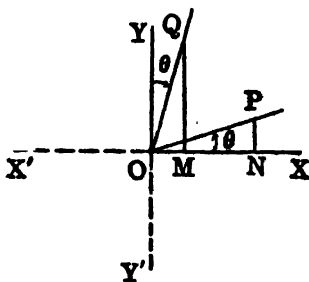


Fig. (i)

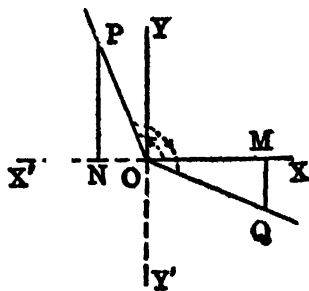


Fig. (ii)

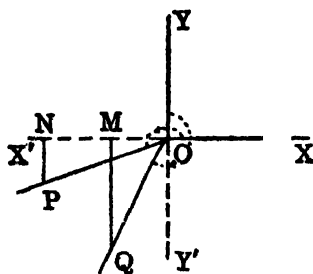


Fig. (iii)

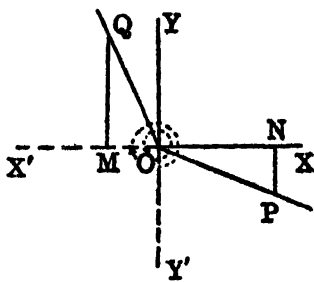


Fig. (iv)

the two right-angled triangles  $PON$ ,  $OQM$  are congruent. The corresponding sides are therefore equal in magnitude. Considering signs as well, we get in all the figures,

$$QM = ON, OM = PN, OQ = OP.$$

Hence, from definition,

$$\sin(90^\circ - \theta) = \sin \angle XOQ = \frac{QM}{OQ} = \frac{ON}{OP} = \cos \theta$$

$$\cos(90^\circ - \theta) = \frac{OM}{OQ} = \frac{PN}{OP} = \sin \theta$$

$$\tan(90^\circ - \theta) = \frac{QM}{OM} = \frac{ON}{PN} = \cot \theta.$$



The reciprocals of these are

$$\operatorname{cosec}(90^\circ - \theta) = \sec \theta,$$

$$\sec(90^\circ - \theta) = \operatorname{cosec} \theta,$$

$$\cot(90^\circ - \theta) = \tan \theta.$$

Obs. The angles  $(90^\circ - \theta)$  is the complement of  $\theta$ , and we derive the result that for a pair of complementary angles, since of one is the cosine of the other and *vice versa*, tangent of one is the cotangent of the other and secant of one is the cosecant of the other. This was verified in the last chapter in connection with the complementary pairs  $30^\circ$  and  $60^\circ$ , as also  $0^\circ$  and  $90^\circ$ .

### 23. Ratios of $(90^\circ + \theta)$ .

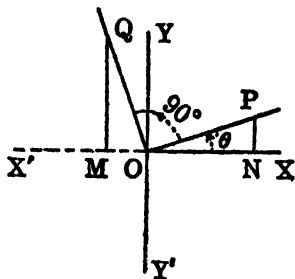


Fig. (i)

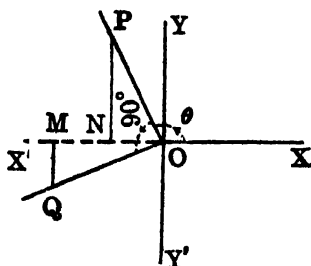


Fig. (ii)

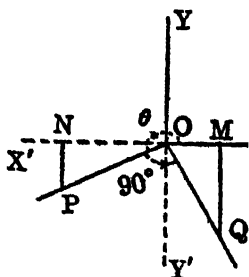


Fig. (iii)

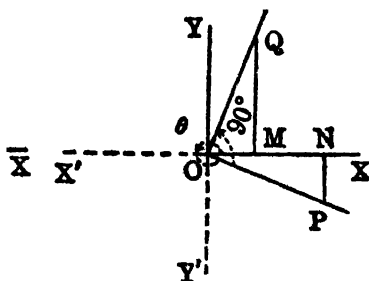


Fig. (iv)

Let a revolving line, starting from  $OX$ , trace out an  $\angle XOP = \theta$ , and further trace out an  $\angle POQ = 90^\circ$ , so that  $\angle XOQ = 90^\circ + \theta$ .

Cut off  $OP = OQ$  along  $OP$  and  $OQ$  respectively and let  $PN, QM$  be perpendiculars on  $OX$  (produced where necessary).

Now,  $OQ$  being perpendicular to  $OP$ , the  $\angle PON =$  the complement of  $\angle QOM = \angle OQM$  in magnitude, and since  $OP = OQ$ , the two right-angled triangles  $OPN$  and  $OQM$  are congruent. The corresponding sides are therefore equal. Considering signs as well, we get, for all the figures,

$$QM = ON, OM = -PN, OQ = OP.$$

Hence, from definition,

$$\sin(90^\circ + \theta) = \sin \angle XOQ = \frac{QM}{OQ} = \frac{ON}{OP} = \cos \theta$$

$$\cos(90^\circ + \theta) = \frac{OM}{OQ} = \frac{-PN}{OP} = -\sin \theta$$

$$\tan(90^\circ + \theta) = \frac{QM}{OM} = \frac{ON}{-PN} = -\cot \theta$$

and considering their reciprocals,

$$\operatorname{cosec}(90^\circ + \theta) = \sec \theta,$$

$$\sec(90^\circ + \theta) = -\operatorname{cosec} \theta,$$

$$\cot(90^\circ + \theta) = -\tan \theta.$$

#### 24. Ratios of $(180^\circ - \theta)$ .

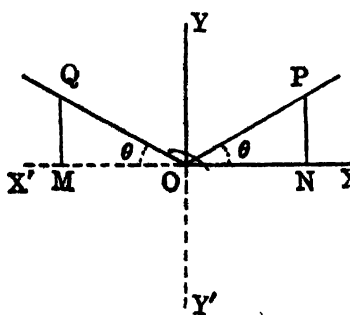


Fig. (i)

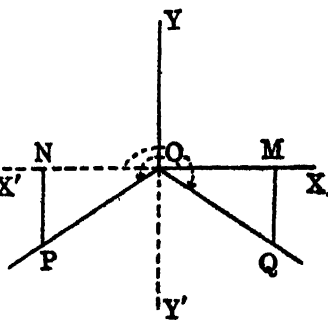


Fig. (ii)

Let  $\angle XOP = \theta$  be traced out by a revolving line, and let another revolving line, starting from  $OX$ , trace out an angle  $180^\circ$  coming up to  $OX'$  and then revolve back and describe an angle  $X'OQ = \theta$ , so that  $\angle XOQ = 180^\circ - \theta$ .

Two figures are given here, one with  $OP$  in the first quadrant and another with  $OP$  in the third quadrant. The two other figures may easily be drawn by the students.

Now cut off  $OP = OQ$ , and draw  $PN$  and  $QM$  perpendiculars on  $OX$  (or  $OX'$  as the case may be). Then  $\angle PON = \angle QOM$  in magnitude, and  $OP = OQ$ . Hence, the right-angled triangles  $PON$  and  $QOM$  are congruent, and so have their corresponding sides equal in magnitude. Taking into consideration the signs, we get for all the figures,

$$QM = PN, OM = -ON, OQ = OP.$$

Hence, for all values of  $\theta$ ,

$$\sin(180^\circ - \theta) = \sin XOQ = \frac{QM}{OQ} = \frac{PN}{OP} = \sin \theta$$

$$\cos(180^\circ - \theta) = \frac{OM}{OP} = \frac{-ON}{OP} = -\cos \theta$$

$$\tan(180^\circ - \theta) = \frac{QM}{OM} = \frac{PN}{-ON} = -\tan \theta$$

and so taking reciprocals,

$$\operatorname{cosec}(180^\circ - \theta) = \operatorname{cosec} \theta,$$

$$\sec(180^\circ - \theta) = -\sec \theta,$$

$$\cot(180^\circ - \theta) = -\cot \theta.$$

**Note.** The first two formulæ may be expressed in the form "*sines of supplementary angles are equal, and cosines of supplementary angles are equal in magnitude but opposite in sign*".

## 25. Ratios of $(180^\circ + \theta)$ .

Let a revolving line starting from  $OX$ , trace out an angle  $XOP = \theta$ , and further trace out an angle  $POQ = 180^\circ$ , so that  $\angle XOQ = 180^\circ + \theta$ .

$OP$  and  $OQ$  are then in one straight line.

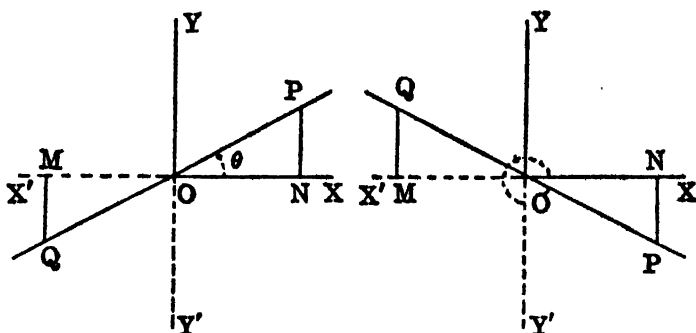


Fig. (i)

Fig. (ii)

Cut off  $OP = OQ$ , and draw  $PN$  and  $QM$  perpendiculars on  $XOX'$ .

Two figures are given here with  $OP$  in the first and fourth quadrants, and the other two may be similarly drawn.

Now,  $POQ$  being a straight line in this case,  $\angle PON = \angle QOM$  in magnitude. Also,  $OP = OQ$ . Hence, the right-angled triangles  $PON$  and  $QOM$  are congruent and have their corresponding sides equal in magnitude. Considering signs, we get in all cases,

$$QM = -PN, OM = -ON, OQ = OP.$$

Thus, for all values of  $\theta$ ,

$$\sin(180^\circ + \theta) = \sin XOQ \therefore \frac{QM}{OQ} = \frac{-PN}{OP} = -\sin \theta$$

$$\cos(180^\circ + \theta) = \frac{OM}{OQ} = \frac{-ON}{OP} = -\cos \theta$$

$$\tan(180^\circ + \theta) = \frac{QM}{OM} = \frac{-PN}{-ON} = \frac{PN}{ON} = \tan \theta$$

and so,

$$\operatorname{cosec} (180^\circ + \theta) = -\operatorname{cosec} \theta,$$

$$\sec (180^\circ + \theta) = -\sec \theta,$$

$$\cot (180^\circ + \theta) = \cot \theta.$$

## 26. Ratios of $(270^\circ - \theta)$ .

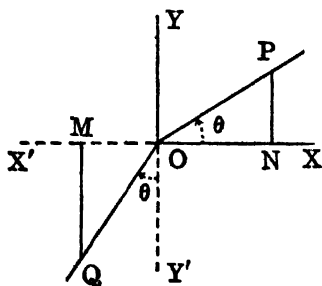


Fig. (i)

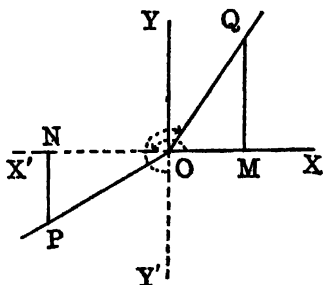


Fig. (ii)

Let  $\angle XOP = \theta$  be traced out by a revolving line, and let another revolving line trace out an angle  $XOY' = 270^\circ$ , thereby coming up to the position  $OY'$ , and then revolve back, tracing out an angle  $Y'OQ = \theta$ , so that  $\angle XOQ = 270^\circ - \theta$ .

Two figures are given here with  $OP$  in the first and third quadrants. The other two may be drawn similarly.

Cut off  $OP = OQ$  and draw  $PN$ ,  $QM$  perpendiculars on  $XOX'$ .

Since  $\angle XOP = \angle Y'OQ$  in magnitude, we easily derive that  $\angle PON = \angle OQM$  in magnitude. Also  $OP = OQ$ . Hence, the two right-angled triangles  $OPN$  and  $OQM$  are congruent. Considering signs, we get for all the figures,

$$QM = -ON, OM = -PN, OQ = OP.$$

Hence, for all values of  $\theta$ ,

$$\sin (270^\circ - \theta) = \sin \angle XOQ = \frac{QM}{OQ} = -\frac{ON}{OP} = -\cos \theta$$

$$\cos (270^\circ - \theta) = \frac{OM}{OQ} = -\frac{PN}{OP} = -\sin \theta ;$$

$$\tan (270^\circ - \theta) = \frac{QM}{OM} = -\frac{ON}{PN} = \frac{ON}{PN} = \cot \theta ;$$

and thus,

$$\operatorname{cosec} (270^\circ - \theta) = -\sec \theta,$$

$$\sec (270^\circ - \theta) = -\operatorname{cosec} \theta,$$

$$\cot (270^\circ - \theta) = \tan \theta.$$

## 27. Ratios of $(270^\circ + \theta)$ .

We may proceed geometrically as in the previous cases. Otherwise we may proceed as follows :

$$\sin (270^\circ + \theta) = \sin (180^\circ + 90^\circ + \theta) = -\sin (90^\circ + \theta) \text{ [from § 25} \\ = -\cos \theta \quad \dots \quad \dots \quad \text{[ from § 23}$$

$$\cos (270^\circ + \theta) = \cos (180^\circ + 90^\circ + \theta) = -\cos (90^\circ + \theta) \\ = -(-\sin \theta) = \sin \theta$$

$$\tan (270^\circ + \theta) = \frac{\sin (270^\circ + \theta)}{\cos (270^\circ + \theta)} = \frac{-\cos \theta}{\sin \theta} = -\cot \theta ;$$

and hence,

$$\operatorname{cosec} (270^\circ + \theta) = -\sec \theta,$$

$$\sec (270^\circ + \theta) = \operatorname{cosec} \theta,$$

$$\cot (270^\circ + \theta) = -\tan \theta.$$

**Note.** The ratios of  $180^\circ - \theta$ ,  $180^\circ + \theta$ ,  $270^\circ - \theta$  can be similarly deduced from the formulæ for ratios of  $90^\circ \pm \theta$ .

## 28. Ratios of $(360^\circ - \theta)$ , $(360^\circ + \theta)$ and $(n. 360^\circ \pm \theta)$ .

It has already been remarked in Art. 2, Chapter I, that angles which differ by complete multiples of  $360^\circ$ , i.e., by an exact number of complete revolutions, have the final positions of the revolving lines coincident, if the initial lines

are the same. Hence, all the trigonometrical ratios of two such angles must be identical in magnitude as well as in sign.

Thus, trigonometrical ratios of  $360^\circ - \theta$  must be same as those of  $-\theta$ . Hence,

$$\sin (360^\circ - \theta) = \sin (-\theta) = -\sin \theta$$

$$\cos (360^\circ - \theta) = \cos (-\theta) = \cos \theta$$

$$\tan (360^\circ - \theta) = \tan (-\theta) = -\tan \theta, \text{ etc.}$$

Trigonometrical ratios of  $360^\circ + \theta$ , or of  $360^\circ \times n \pm \theta$ , where  $n$  is an integer, positive or negative, must similarly be same as those of  $\theta$ , or of  $\pm \theta$ .

Thus, *in determining trigonometrical ratios of angles, complete multiples of  $360^\circ$  (i.e.,  $2\pi$ ) may be always added or subtracted.*

29. All the above results may, for easy remembrance, be summed up in a simple rule.

If  $\theta$  be associated with an even multiple of  $90^\circ$  by + or - sign, (e.g.,  $180^\circ - \theta$ ,  $180^\circ + \theta$ ,  $360^\circ - \theta$ ,  $360^\circ + \theta$ , etc.) the ratio is not altered in form (i.e., sine remains sine, cosine remains cosine, etc.). To determine the sign, assuming  $\theta$  to be acute, find out the quadrant in which the associated angle lies, and determine the sign according to the rule, "all, sin, tan, cos".

If  $\theta$  be associated with an odd multiple of  $90^\circ$  by + or - sign, (e.g.,  $90^\circ - \theta$ ,  $90^\circ + \theta$ ,  $270^\circ - \theta$ ,  $270^\circ + \theta$ , etc.) the ratio is altered (sine becomes cosine, cosine becomes sine, tangent becomes cotangent, etc.). Moreover, the sign of the result is determined as in the previous paragraph.

**Example.** Consider formulae for  $\tan (270^\circ - \theta)$  and  $\sec (180^\circ + \theta)$ .

$$270^\circ - \theta = 3.90^\circ - \theta \text{ (multiple of } 90^\circ \text{ is odd).}$$

Hence, the ratio will be altered,  $\tan$  changing into  $\cot$ . Moreover,  $\theta$  being assumed acute (whether it is actually so or not, it does not matter),  $270^\circ - \theta$  falls in the third quadrant, where  $\tan$  is positive.

Hence,  $\tan (270^\circ - \theta) = +\cot \theta$ .

$180^\circ + \theta$  has got  $\theta$  associated with even multiple of  $90^\circ$ . Hence, the ratio does not alter in form,  $\sec$  remaining  $\sec$ . Also,  $180^\circ + \theta$  falls in the third quadrant, if  $\theta$  be assumed acute, where  $\sec$  (by the rule "all, sin, tan, cos") is negative.

Hence,  $\sec (180^\circ + \theta) = -\sec \theta$ .

N. B. The angle  $-\theta$  may be written as  $0^\circ 360^\circ - \theta$ , and 0 may be considered *even* in applying the above rule.

Thus,  $\theta$  being supposed acute,  $-\theta$  falls in the fourth quadrant, where  $\cos$  and  $\sec$  only are positive. The form of the ratio not changing in this case,  $\sin (-\theta) = -\sin \theta$ ,  $\cos (-\theta) = +\cos \theta$ , etc.

### 30. Special angles (*outside the first quadrant*).

In Art. 24, putting  $\theta = 60^\circ, 45^\circ, 30^\circ$  and  $0^\circ$  respectively we can deduce the following results :

$$\sin 120^\circ = \sin 60^\circ = \frac{\sqrt{3}}{2} ; \quad \cos 120^\circ = -\cos 60^\circ = -\frac{1}{2} .$$

$$\sin 135^\circ = \sin 45^\circ = \frac{1}{\sqrt{2}} ; \quad \cos 135^\circ = -\cos 45^\circ = -\frac{1}{\sqrt{2}} .$$

$$\sin 150^\circ = \sin 30^\circ = \frac{1}{2} ; \quad \cos 150^\circ = -\cos 30^\circ = -\frac{\sqrt{3}}{2} .$$

$$\sin 180^\circ = \sin 0^\circ = 0 ; \quad \cos 180^\circ = -\cos 0^\circ = -1 .$$

And similarly from Arts. 27 and 28, putting  $\theta = 0$ ,

$$\sin 270^\circ = -\cos 0^\circ = -1 ; \quad \cos 270^\circ = \sin 0^\circ = 0 ;$$

$$\sin 360^\circ = \sin 0^\circ = 0 ; \quad \cos 360^\circ = \cos 0^\circ = 1 .$$

From the above we get,

$$\tan 180^\circ = 0 ; \tan 270^\circ = \pm \infty ; \tan 360^\circ = 0 .$$



**Examples worked out.****Ex. 1.** Find the value of  $\cot (-1575^\circ)$ .

$$\begin{aligned}\cot (-1575^\circ) &= -\cot (1575^\circ) = -\cot (4 \times 360^\circ + 185^\circ) \\ &= -\cot (185^\circ) = -\cot (180^\circ - 45^\circ) \\ &= \cot 45^\circ = 1.\end{aligned}$$

**Ex. 2.** Find the value of  $\cot \theta - \tan \theta$ , where  $\theta = \frac{17\pi}{3}$ .

$\frac{17\pi}{3} = 6\pi - \frac{\pi}{3}$ , and omitting complete multiples of  $360^\circ$  i.e., of  $2\pi$ , whereby trigonometrical ratios are not altered, we get,

$$\begin{aligned}\cot \frac{17\pi}{3} &= \cot \left( -\frac{\pi}{3} \right) = -\cot \frac{\pi}{3} = -\cot 60^\circ = -\frac{1}{\sqrt{3}}. \\ \tan \frac{17\pi}{3} &= \tan \left( -\frac{\pi}{3} \right) = -\tan \frac{\pi}{3} = -\tan 60^\circ = -\sqrt{3}. \\ \therefore \cot \theta - \tan \theta &= -\frac{1}{\sqrt{3}} + \sqrt{3} = \frac{3-1}{\sqrt{3}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}.\end{aligned}$$

**Ex. 3.** Prove that

$$\sin (420^\circ) \cos (390^\circ) + \cos (-300^\circ) \sin (-330^\circ) = 1. \quad [H. S. 1962]$$

$$\begin{aligned}\text{L. H. side} &= \sin (360^\circ + 60^\circ) \cos (360^\circ + 30^\circ) \\ &\quad + \cos (-360^\circ + 60^\circ) \sin (-360^\circ + 30^\circ) \\ &= \sin 60^\circ \cos 30^\circ + \cos 60^\circ \sin 30^\circ \\ &= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4} + \frac{1}{4} = 1.\end{aligned}$$

**Ex. 4.** Express  $\cot (-1358^\circ)$  in terms of the ratio of a positive angle less than  $45^\circ$ .

$$\begin{aligned}\cot (-1358^\circ) &= \cot (-4 \times 360^\circ + 82^\circ) \\ &= \cot 82^\circ = \cot (90^\circ - 8^\circ) \\ &= \tan 8^\circ.\end{aligned}$$

**Note.** Ratios of angles of any magnitude and sign can always be expressed in terms of a ratio of a positive angle less than  $45^\circ$ .

Ex. 5. Express

$\frac{\cos(90^\circ + \theta) \sec(-\theta) \tan(180^\circ - \theta)}{\sec(360^\circ + \theta) \sin(180^\circ + \theta) \cot(90^\circ - \theta)}$  in its simplest form.

The given expression

$$\begin{aligned} &= \frac{-\sin \theta \cdot \sec \theta \cdot (-\tan \theta)}{\sec \theta \cdot (-\sin \theta) \cdot \tan \theta} \\ &= -1. \end{aligned}$$

### Examples IV

1. Write down the values of  $\sin 150^\circ$ ,  $\cot 840^\circ$ ,  $\operatorname{cosec}(-660^\circ)$  and  $\tan(-1125^\circ)$ .

2. Find the values of  $\sin\left(-\frac{11\pi}{4}\right)$ ,  $\operatorname{cosec}\left(\frac{16\pi}{3}\right)$ ,

$$\tan\left(\frac{3\pi}{2} + \frac{\pi}{3}\right) \text{ and } \cos\left(\frac{5\pi}{2} - \frac{19\pi}{3}\right).$$

3. Evaluate  $\sin\left(-1230^\circ\right) - \cos\left\{\left(2n+1\right)\pi + \frac{\pi}{3}\right\}$ , where  $n$  is a negative integer.

4. Find the value of  $\sin\left\{n\pi + (-1)^n \frac{\pi}{3}\right\}$ , where  $n$  is any integer.

5. Find all the values of

$$(i) \tan\left\{\frac{n\pi}{2} + (-1)^n \frac{\pi}{4}\right\};$$

$$(ii) \operatorname{cosec}\left\{\frac{n\pi}{2} + (-1)^n \frac{\pi}{6}\right\},$$

where  $n$  is any integer.

6. Show that  $\cos\left(2m\pi \pm \frac{\pi}{3}\right)$  and  $\tan\left(m\pi + \frac{\pi}{6}\right)$  have one value each for all integral values of  $m$ .

7. Prove that,  $n$  being any integer

$$(i) \cos (n\pi + a) = (-1)^n \cos a.$$

$$(ii) \tan (n\pi - a) = -\tan a.$$

8. Prove that

$$(i) \cos \theta = -\cos (\theta - 180^\circ).$$

$$(ii) \tan \theta = -\cot (\theta - \frac{3}{2}\pi).$$

9. Prove that

$$(i) \sin (780^\circ) \cos (390^\circ) - \sin (330^\circ) \cos (-300^\circ) = 1.$$

$$(ii) \cos 306^\circ + \cos 234^\circ + \cos 162^\circ + \cos 18^\circ = 0.$$

$$(iii) \frac{\sin 250^\circ + \tan 290^\circ}{\cot 200^\circ + \cos 340^\circ} = -1.$$

10. Simplify

$$\frac{\sin^3(\pi + \theta) \cdot \tan(2\pi - \theta) \cdot \sec^2(\pi - \theta)}{\cos^2(\frac{1}{2}\pi + \theta) \cdot \operatorname{cosec}^2 \theta \cdot \sin(\pi - \theta)}$$

and determine its value when  $\theta = 225^\circ$ .

11. Prove that

$$\begin{aligned} \sin(\frac{1}{2}\pi + \theta) \cos(\pi - \theta) \cot(\frac{3}{2}\pi + \theta) \\ = \sin(\frac{1}{2}\pi - \theta) \sin(\frac{3}{2}\pi - \theta) \cot(\frac{1}{2}\pi + \theta). \end{aligned}$$

12. Evaluate

$$(i) \sin^2 \frac{\pi}{4} + \sin^2 \frac{3\pi}{4} + \sin^2 \frac{5\pi}{4} + \sin^2 \frac{7\pi}{4}.$$

$$(ii) \cot \frac{\pi}{20} \cot \frac{3\pi}{20} \cot \frac{5\pi}{20} \cot \frac{7\pi}{20} \cot \frac{9\pi}{20}.$$

$$(iii) \sin x + \sin(\pi + x) + \sin(2\pi + x) + \dots \text{ to } n \text{ terms.}$$

13. If  $\tan \theta = \frac{5}{12}$  and  $\cos \theta$  is negative, find the value of

$$\frac{\sin \theta + \cos(-\theta)}{\sec(-\theta) + \tan \theta}.$$

14. An angle  $\theta$  lies between  $180^\circ$  and  $270^\circ$ , and  $\operatorname{cosec} \theta = -\frac{7}{5}$ . Find  $\cot \theta$ .

15. Express in terms of ratios of positive angles less than  $45^\circ$  :

$$(i) \cot(-1054^\circ). \quad (ii) \sin(1145^\circ).$$

$$(iii) \sec(-1491^\circ). \quad (iv) \cos \frac{35\pi}{9}.$$

16. Find the values of  $\theta$ , when

$$(i) \tan \theta = -\sqrt{3} \text{ and } \theta \text{ lies between } 270^\circ \text{ and } 360^\circ.$$

$$(ii) \cos \theta = -\frac{1}{2}, \text{ and } 450^\circ < \theta < 540^\circ.$$

17. Solve for  $\theta$ , giving all the possible values, when  $0^\circ < \theta < 360^\circ$  :

$$(i) \cos \theta + \sqrt{3} \sin \theta = 2. \quad [C. U. 1936]$$

$$(ii) 2 \sin^2 \theta + 3 \cos \theta = 0.$$

$$(iii) 3(\sec^2 \theta + \tan^2 \theta) = 5.$$

$$(iv) \cot \theta + \tan \theta = 2 \sec \theta.$$

$$(v) 1 - 2 \sin \theta - 2 \cos \theta + \cot \theta = 0.$$

18. If  $A, B, C$  be the angles of a triangle, show that  $\sin(A+B) - \cos C = \cos(A+B) + \sin C$ .

19. If  $A, B, C$  be the angles of a triangle, show that 
$$\frac{\tan(B+C) + \tan(C+A) + \tan(A+B)}{\tan(\pi-A) + \tan(2\pi-B) + \tan(3\pi-C)} = 1.$$

20. If  $A, B, C, D$  be the angles of a quadrilateral, show that

$$\cos \frac{1}{2}(A+C) + \cos \frac{1}{2}(B+D) = 0.$$

If the quadrilateral be cyclic, then

$$\cos A + \cos B + \cos C + \cos D = 0.$$

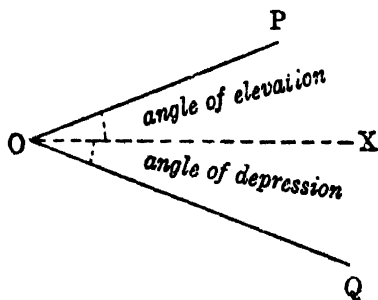

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CHAPTER V  
SIMPLE PRACTICAL APPLICATIONS OF  
TRIGONOMETRY

(*Heights and Distances*)

31. One of the most important applications of Trigonometry is in the determination of *heights and distances* of distant objects which are not directly measurable, by observations of angles subtended by those objects at the eye of the observer. These angles may be measured by instruments known as Sextants or Theodolites or by other angle-measuring instruments. Thus, Trigonometry plays a very important part in *land survey*. It is also extensively used by Astronomers in determining the distances of the heavenly bodies like the sun, the moon and the stars.

Two angles are very often used in the practical applications of Trigonometry, and they are defined as follows :

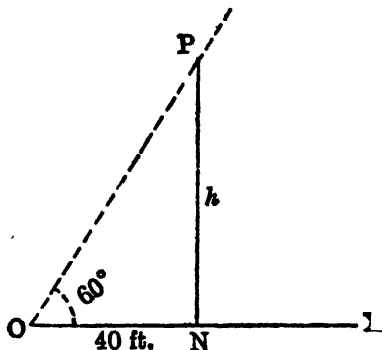


If a horizontal line  $OX$  be drawn through  $O$ , the eye of an observer, the angle which the line joining  $O$  to a point  $P$  above  $OX$  makes with  $OX$  is called the **Angle of Elevation** or *altitude of  $P$*  as seen from  $O$ .

If  $Q$  be below the horizontal line  $OX$ , the angle  $XOQ$  measured below  $OX$  is called the **Angle of Depression** of  $Q$  as seen from  $O$ .

### 32. Illustrative Examples.

**Ex. 1.** *From a distance of 40 feet from the foot of a palm tree in a horizontal field, the angle of elevation of the top of the tree is observed to be  $60^\circ$ . Find the height of the tree.*



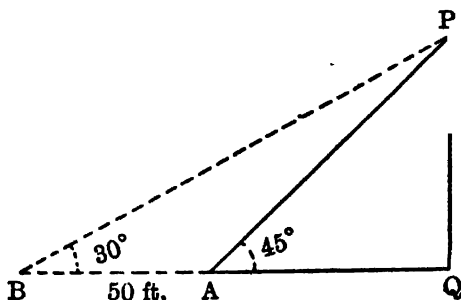
Let  $h$  ft. be the height of the tree  $PN$ , and  $\angle NOP$ , the angle of elevation of  $P$  as seen from  $O$ , where  $ON = 40$  ft., is  $60^\circ$ .

$$\text{Then, } \frac{h}{40} = \tan PON = \tan 60^\circ = \sqrt{3} ;$$

$$\therefore h = 40 \sqrt{3} \text{ ft.} = 69.28 \dots \text{ ft.}$$

**Ex. 2.** *From one bank of a river, the top of a building just on the opposite bank is observed to have an elevation of  $45^\circ$ . On receding 50 ft. from the bank, perpendicular to its edge, the angle of elevation becomes  $30^\circ$ . Find the breadth of the river and the height of the building.*

$AQ$  being the breadth of the river,  $PQ$  the height of the building,  $\angle PAQ = 45^\circ$ . Also,  $AB$  being 50 ft.  $\angle PBQ = 30^\circ$ .



$$\text{Now, } \frac{BQ}{PQ} = \cot 30^\circ, \frac{AQ}{PQ} = \cot 45^\circ.$$

$$\text{Hence, subtracting, } \frac{AB}{PQ} = \cot 30^\circ - \cot 45^\circ,$$

$$\text{or, } \frac{50}{PQ} = \sqrt{3} - 1;$$

$$PQ = \frac{50}{\sqrt{3} - 1} = \frac{50(\sqrt{3} + 1)}{2} = 68.3 \text{ ft. nearly.}$$

$$\text{Also, } \frac{AQ}{PQ} = \cot 45^\circ = 1; \therefore AQ = PQ = 68.3 \text{ ft.}$$

Thus, the breadth of the river and the height of the building are both 68.3 ft. nearly.

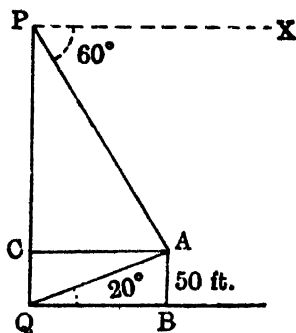
**Ex. 3.** *The angles of depression and elevation of the top of a tower 50 ft. high from the top and bottom of a second tower are  $60^\circ$  and  $20^\circ$  respectively. Find the height of the second tower to the nearest foot. [ Given  $\cot 20^\circ = 2.747$ . ]*

$PQ$  is the second tower, and  $\angle XPA = 60^\circ$ ,  $\angle BQA = 20^\circ$ ,  $AB = 50$  ft.,  $AC$  is parallel to  $BQ$  or  $PX$ , so that  $\angle PAC =$  the alternate angle  $\angle XPA = 60^\circ$ .

Now,  $\frac{QB}{AB} = \cot 20^\circ, \therefore QB = AB \cot 20^\circ.$

Also,  $\frac{PC}{CA} = \tan PAC = \tan 60^\circ;$

$$PC = CA \tan 60^\circ = QB \tan 60^\circ \\ = AB \cot 20^\circ \tan 60^\circ.$$



$$\therefore \text{height } PQ = PC + CQ = PC + AB \\ = AB (\cot 20^\circ \tan 60^\circ + 1) \\ = 50 (2.747 \times \sqrt{3} + 1) \\ = 287.8 \dots \text{ ft.} = 288 \text{ ft. nearly.}$$

**Ex. 4.** *The elevation of a hill from a place P due East of it is  $45^\circ$ , and at a place Q due South of P, the elevation is  $30^\circ$ . If the distance PQ be 400 yds., find the height of the hill.*

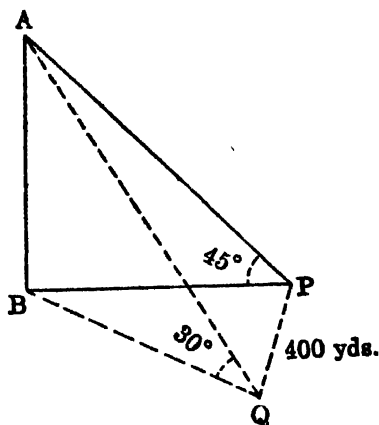
A is the top of the hill, B is the point vertically below it on the ground. BP is due East, PQ is due South, so that BPQ is a right angle. Also ABP and ABQ are both right angles.

Now,  $\frac{BQ}{AB} = \cot AQB = \cot 30^\circ = \sqrt{3},$

and  $\frac{BP}{AB} = \cot APB = \cot 45^\circ = 1.$



Hence,  $BQ = AB \sqrt{3}$ ,  $BP = AB$ ,



and  $PQ^2 = BQ^2 - BP^2 = AB^2(3 - 1) = 2AB^2$ .

$\therefore AB = \frac{PQ}{\sqrt{2}} = \frac{400}{\sqrt{2}} = 200\sqrt{2} = 283$  yds. nearly.

### Examples V

1. From the top of a tower by the seaside, 100 feet high, it was observed that the angle of depression of the bottom of a ship at anchor was  $30^\circ$ . Find the distance of the ship from the bottom of the tower.

2. Two straight roads, which cross one another, meet a river with straight course at angles  $60^\circ$  and  $30^\circ$  respectively. If it be 3 miles by the longer of the two roads, from the crossing to the river, how far is it by the shortest? If there be a foot-path which goes the shortest way from the crossing to the river, what is the distance by it?

3. Two poles are of equal height; a person standing midway between the line joining their bases observes the

elevation of the poles to be  $30^\circ$ . After walking 40 feet towards one of them, he observes that the same pole now subtends an angle of  $60^\circ$ . Find their height and the distance between them.

4. A straight palm tree 60 feet high, is broken by the wind but not completely separated, and its upper part meets the ground at an angle of  $30^\circ$ . Find the distance of the point where the top of the tree meets the ground, from the root, and also the height at which the tree is broken.

5. Two posts are 120 ft. apart, and the height of one is double that of the other. From the middle point of the line joining their feet, an observer finds the angular elevations of their tops to be complementary. Find the height of the shorter post. [ *H. S. 1961 Com.* ]

6. The Bally bridge subtends an angle of  $45^\circ$  at a given point at the edge of the river ; 800 yds. higher up, it subtends an angle of  $30^\circ$ . The course of the river here is straight and perpendicular to the bridge. Find the length of the bridge.

7. The height of a house subtends a right angle at an opposite window, the top being  $60^\circ$  above a horizontal straight line through the window ; find the height of the house, taking the breadth of the street to be 30 feet.

8. From an aeroplane vertically over a straight road, the angles of depression of two consecutive milestones are observed to be  $45^\circ$  and  $60^\circ$  ; find the height of the aeroplane.

9. From a ship sailing due South-East at the rate of 5 miles an hour, a light-house is observed to be  $30^\circ$  North of East, and after 4 hours, it is seen due North ; find the distance of the light-house from the final position of the ship.

10. The shadow of a tower standing on a level plane is found to be 40 feet longer when the sun's altitude is  $45^\circ$  than when it is  $60^\circ$ . Find the height of the tower.

11. From the lower window of a house the angular elevation of a church-steeple is found to be  $45^\circ$  and from a window 20 feet above, the elevation is  $30^\circ$ . How far is the church from the house ?

12. A light-house facing East sends out a fan-shaped beam of light extending from S. E. to N. E. An observer sailing due North, after meeting the light continues to see it for  $10\sqrt{2}$  minutes. When leaving the fan of light, the ship is 10 miles from the light-house. Find the speed of the ship.

13. A pole 100 ft. high stands vertically at the centre of a horizontal equilateral triangle, each side of which subtends an angle of  $60^\circ$  at the top of the pole. Find the side of the triangle.

14. Two chimneys are of equal height. A person standing between them in the line joining their bases observes the elevation of the nearer one to be  $60^\circ$ . After walking 80 feet in a direction at right angles to the line joining their bases, he observes the elevations of the two to be  $45^\circ$  and  $30^\circ$  respectively. Find the height and the distance between them.

15. At the foot of a mountain the elevation of its summit is  $45^\circ$ ; after ascending 1 mile towards the mountain up an incline of  $30^\circ$ , the elevation changes to  $60^\circ$ . Find the height of the mountain.

16. From a station, two light-houses  $A$  and  $B$  are seen in directions North and  $30^\circ$  East of North respectively; if  $A$  were one-third as far off as it really is, it would appear due West of  $B$ . If the distance of  $B$  from the station be 10 miles, find the distance of  $B$  from  $A$ .

17. A person walking along a straight road observes a tall tree standing in front of a tower, both being on the road before him. The elevation of the top of the tower is  $45^\circ$ , and that of the top of the tree  $30^\circ$ ; on advancing 100 feet he finds the tower and the tree to have the same elevation  $60^\circ$ ; supposing the height of the eye of the man to be 5 feet, find the height of the tower and of the tree.

18. A man on the top of a rock rising on a seashore, observes a boat coming towards it at an angle of depression  $30^\circ$ ; 10 minutes later the angle of depression is  $60^\circ$ . The height of the rock being 4000 feet, find the speed of the boat in miles per hour.

19. A person walking along a straight level road observes the elevation of the top of a hill to be  $60^\circ$  when he is nearest the hill, and after walking 200 yards in a direction perpendicular to the direction of the hill from this point, observes the elevation to be  $30^\circ$ . Find the approximate height of the hill.

20. A square tower stands on a horizontal plane. From a point in this plane, only three of its upper corners are visible, and their angles of elevation are  $45^\circ, 60^\circ, 45^\circ$ . Find the ratio of the height of the tower to its breadth.

21. Two wheels, the sum of whose radii is 10 feet, are placed flatly on a table with their centres at a distance of 20 ft. An endless string, quite stretched, is partly wrapped round the wheels and crosses itself between them. Show that the length of the string is nearly 76.5 feet.

22. On a still day, from a station *A* an airship is observed due North at an elevation of  $60^\circ$ , while from a station *B* it is observed due East at an elevation of  $45^\circ$ . At this instant of observation, a parachute message is dropped from the airship, and the observer at *A* has to walk a mile to reach the message. Find the distance between the two stations.

23. From the foot of a column the angle of elevation of the top of a tower is  $45^\circ$  and from the top of the column the angle of depression of the bottom of the tower is  $30^\circ$ . A man walks 10 ft. from the bottom of the column towards the tower and notices the angle of elevation of its top to be  $60^\circ$ . Find the height of the column.

# CHAPTER VI

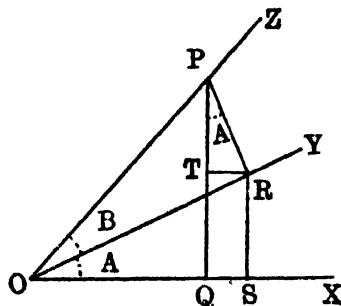
## COMPOUND ANGLES

33. To Prove that

$$\sin (A+B)=\sin A \cos B+\cos A \sin B$$

$$\cos (A+B)=\cos A \cos B-\sin A \sin B,$$

when  $A$  and  $B$  are positive and acute and  $(A+B) < 90^\circ$ .



Let a revolving line starting from the position  $OX$  trace out an angle  $XOY=A$  and then revolving further, trace out an angle  $YOZ=B$ ; then  $\angle XOZ=A+B$ .

In  $OZ$ , the bounding line of the compound angle  $A+B$ , take any point  $P$  and draw  $PQ$  and  $PR$  perpendiculars to  $OX$  and  $OY$  respectively; also draw  $RS$  and  $RT$  perpendiculars to  $OX$  and  $PQ$  respectively.

From the right-angled  $\triangle POQ$ ,

$$\begin{aligned}\sin (A+B) &= \frac{PQ}{OP} = \frac{QT+TP}{OP} = \frac{RS+PT}{OP} = \frac{RS}{OP} + \frac{PT}{OP} \\ &= \frac{RS}{OR} \cdot \frac{OR}{OP} + \frac{PT}{PR} \cdot \frac{PR}{OP} \\ &= \sin A \cos B + \cos A \sin B.\end{aligned}$$

Now,  $\angle TPR = 90^\circ - \angle TRP = \angle TRO = \angle ROS = A$ .

$$\therefore \sin (A+B)=\sin A \cos B+\cos A \sin B.$$

Again,

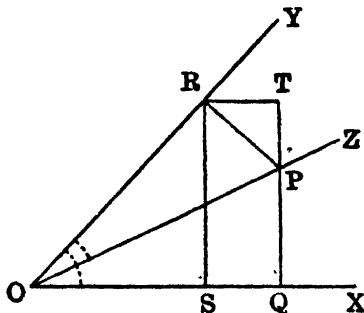
$$\begin{aligned}\cos (A+B) &= \frac{OQ}{OP} = \frac{OS - QS}{OP} = \frac{OS - TR}{OP} = \frac{OS}{OP} - \frac{TR}{OP} \\ &= \frac{OS}{OR} \cdot \frac{OR}{OP} - \frac{TR}{PR} \cdot \frac{PR}{OP} \\ &= \cos A \cos B - \sin A \sin B \\ &= \cos A \cos B - \sin A \sin B.\end{aligned}$$

34. To prove that

$$\sin (A-B)=\sin A \cos B-\cos A \sin B$$

$$\cos (A-B)=\cos A \cos B+\sin A \sin B,$$

when  $A$  and  $B$  are positive and acute, and  $A > B$ .



Let a revolving line start from the position  $OX$  and trace out an angle  $XOY=A$  and then revolving back trace out an angle  $YOZ=B$ ; then  $\angle XOZ=A-B$ .

In  $OZ$ , the bounding line of the compound angle  $A-B$ , take any point  $P$ , and draw  $PQ$  and  $PR$  perpendiculars to  $OX$  and  $OY$  respectively; and draw  $RS$  and  $RT$  perpendiculars to  $OX$  and  $QP$  produced respectively.

From the right-angled  $\triangle POQ$ ,

$$\sin (A-B)=\frac{PQ}{OP}=\frac{TQ-PT}{OP}=\frac{RS-PT}{OP}=\frac{RS}{OP}-\frac{PT}{OP}$$

$$= \frac{RS}{OR} \cdot \frac{OR}{OP} - \frac{PT}{PR} \cdot \frac{PR}{OP}$$

$$= \sin A \cos B - \cos TPR \cdot \sin B.$$

But  $\angle TPR = 90^\circ - \angle TRP = \angle YRT = \angle YOX = A$ .

$$\therefore \sin(A - B) = \sin A \cos B - \cos A \sin B.$$

Again,

$$\cos(A - B) = \frac{OQ}{OP} = \frac{OS + SQ}{OP} = \frac{OS + RT}{OP} = \frac{OS}{OP} + \frac{RT}{OP}$$

$$= \frac{OS}{OR} \cdot \frac{OR}{OP} + \frac{RT}{RP} \cdot \frac{RP}{OP}$$

$$= \cos A \cos B + \sin TPR \cdot \sin B$$

$$= \cos A \cos B + \sin A \sin B.$$

**Obs.** In the above Geometrical proofs, it is assumed that the angles  $A$ ,  $B$ ,  $A+B$  are all less than a right angle and that  $A-B$  is positive. If the angles are not so restricted, the same method of proof (there being some modifications in the figures) will apply, due attention being paid to the signs of the quantities involved.\*

Thus, the above formulæ are perfectly general.

**Note 1.** The sum or difference of two or more angles is called a *Compound angle*; such as  $A+B$ ,  $A-B$ ,  $A+B+C$  etc.

The expansions  $\sin(A \pm B)$  and  $\cos(A \pm B)$  are generally called the "*Addition formulæ or Addition and Subtraction Theorems*".

**Note 2.** Assuming the truth of the above formulæ for acute angles, they can be shown to be true for angles of any magnitude, as follows :

Let us consider  $\sin(A+B)$ .

Let  $A$  and  $B$  be acute and  $A+B < 90^\circ$ .

Let  $A_1 = 90^\circ + A$ ;  $B_1 = B$ .

$$\begin{aligned} \text{Now, } \sin(A_1 + B_1) &= \sin\{(90^\circ + A) + B\} = \sin\{90^\circ + (A+B)\} \\ &= \cos(A+B) = \cos A \cos B - \sin A \sin B \quad [\text{by Art. 33}] \\ &= \sin(90^\circ + A) \cos B + \cos(90^\circ + A) \sin B \\ &= \sin A_1 \cos B_1 + \cos A_1 \sin B_1. \end{aligned}$$

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\*See *Appendix, Arts. 2-4*. Also for alternative proofs of *Arts. 33* and *34*, see *Appendix, Art. 5*.

Again, let  $A_2 = -A$ ,  $B_2 = B$ .

$$\begin{aligned}\text{Then, } \sin(A_2 + B_2) &= \sin(-A + B) = -\sin(A - B) \\ &= -\sin A \cos B + \cos A \sin B, \quad [\text{by Art. 34}] \\ &= \sin(-A) \cos B + \cos(-A) \sin B \\ &= \sin A_2 \cos B_2 + \cos A_2 \sin B_2.\end{aligned}$$

Thus, the above formulæ remain true if any of the two angles is either increased by  $90^\circ$ , or has its sign changed.

In the same way it may be shown that the other three formulæ for  $\cos(A+B)$ ,  $\sin(A-B)$  and  $\cos(A-B)$  will continue to hold good unchanged in form, if any of the two angles be either increased by  $90^\circ$  or has its sign changed.

Now starting from positive acute-angled values of  $A$  and  $B$ , combining the two processes of increasing one of the angles by  $90^\circ$ , and reversing the sign of any one, we can arrive at values of  $A$  and  $B$  of any magnitude, positive or negative and the four formulæ will still hold good.

Thus, the formulæ for  $\sin(A \pm B)$  and  $\cos(A \pm B)$  are perfectly general.

**35. Ex. 1.** Find the values of

$\sin 75^\circ$ ,  $\cos 75^\circ$ ,  $\sin 15^\circ$  and  $\cos 15^\circ$ .

$$\sin 75^\circ = \sin(45^\circ + 30^\circ) = \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}.$$

$$\cos 75^\circ = \cos(45^\circ + 30^\circ) = \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}.$$

$\sin 15^\circ = \sin(45^\circ - 30^\circ) = \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ$   
and  $\cos 15^\circ = \cos(45^\circ - 30^\circ) = \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ$ ;  
therefore, substituting the values of  $\sin 45^\circ$ ,  $\cos 45^\circ$  etc.  
as before, we get



$$\sin 15^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}} \quad \text{and} \quad \cos 15^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}.$$

**Note.** The values of  $\sin 15^\circ$  and  $\cos 15^\circ$  can also be deduced from the fact that

$$\begin{aligned} \sin 15^\circ &= \sin (90^\circ - 75^\circ) = \cos 75^\circ \\ \text{and} \quad \cos 15^\circ &= \cos (90^\circ - 75^\circ) = \sin 75^\circ. \end{aligned}$$

**Ex. 2.** Show that

$$\begin{aligned} \text{(i)} \quad \sin (A+B) \sin (A-B) &= \sin^2 A - \sin^2 B \\ &= \cos^2 B - \cos^2 A. \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad \cos (A+B) \cos (A-B) &= \cos^2 A - \sin^2 B \\ &= \cos^2 B - \sin^2 A. \end{aligned}$$

(i) Left side

$$\begin{aligned} &= (\sin A \cos B + \cos A \sin B)(\sin A \cos B - \cos A \sin B) \\ &= \sin^2 A \cos^2 B - \cos^2 A \sin^2 B \\ &= \sin^2 A (1 - \sin^2 B) - (1 - \sin^2 A) \sin^2 B \\ &= \sin^2 A - \sin^2 B \\ &= (1 - \cos^2 A) - (1 - \cos^2 B) = \cos^2 B - \cos^2 A. \end{aligned}$$

(ii) Left side

$$\begin{aligned} &= (\cos A \cos B - \sin A \sin B)(\cos A \cos B + \sin A \sin B) \\ &= \cos^2 A \cos^2 B - \sin^2 A \sin^2 B \\ &= \cos^2 A (1 - \sin^2 B) - (1 - \cos^2 A) \sin^2 B \\ &= \cos^2 A - \sin^2 B \\ &= (1 - \sin^2 A) - (1 - \cos^2 B) = \cos^2 B - \sin^2 A. \end{aligned}$$

**Note.** The results of Ex. 1 and Ex. 2 are very useful and should be carefully remembered.

**36.** To prove that

$$\text{(i)} \quad \tan (A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

$$\text{(ii)} \quad \tan (A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}.$$

We have

$$\tan (A+B)=\frac{\sin (A+B)}{\cos (A+B)}=\frac{\sin A \cos B+\cos A \sin B}{\cos A \cos B-\sin A \sin B}$$

Now, dividing the numerator and denominator by  $\cos A \cos B$ , we have,

$$\begin{aligned}\tan (A+B) &= \frac{\frac{\sin A \cos B}{\cos A \cos B} + \frac{\cos A \sin B}{\cos A \cos B}}{\frac{\cos A \cos B}{\cos A \cos B} - \frac{\sin A \sin B}{\cos A \cos B}} \\ &= \frac{\tan A + \tan B}{1 - \tan A \tan B}\end{aligned}$$

Again,

$$\tan (A-B)=\frac{\sin (A-B)}{\cos (A-B)}=\frac{\sin A \cos B-\cos A \sin B}{\cos A \cos B+\sin A \sin B}$$

Now, dividing the numerator and denominator by  $\cos A \cos B$ , we have, as before,

$$\tan (A-B)=\frac{\tan A - \tan B}{1 + \tan A \tan B}$$

37. To prove that

$$(i) \cot (A+B)=\frac{\cot A \cot B-1}{\cot B+\cot A}$$

$$(ii) \cot (A-B)=\frac{\cot A \cot B+1}{\cot B-\cot A}$$

$$(i) \cot (A+B)=\frac{\cos (A+B)}{\sin (A+B)}=\frac{\cos A \cos B-\sin A \sin B}{\sin A \cos B+\cos A \sin B}$$

Now, dividing the numerator and denominator by  $\sin A \sin B$ , we have,

$$\begin{aligned}\cot (A+B) &= \frac{\frac{\cos A \cos B}{\sin A \sin B} - \frac{\sin A \sin B}{\sin A \sin B}}{\frac{\sin A \cos B}{\sin A \sin B} + \frac{\cos A \sin B}{\sin A \sin B}} \\ &= \frac{\cot A \cot B - 1}{\cot B + \cot A}\end{aligned}$$

$$(ii) \cot(A-B) = \frac{\cos(A-B)}{\sin(A-B)} = \frac{\cos A \cos B + \sin A \sin B}{\sin A \cos B - \cos A \sin B}.$$

Now, dividing the numerator and denominator by  $\sin A \sin B$ , we have, as before,

$$\cot(A-B) = \frac{\cot A \cot B + 1}{\cot B - \cot A}.$$

38. Ex. 1. Find the values of  $\tan 75^\circ$  and  $\tan 15^\circ$ .

$$\begin{aligned} \tan 75^\circ &= \tan(45^\circ + 30^\circ) = \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ} \\ &= \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}} = \frac{\sqrt{3} + 1}{\sqrt{3} - 1} = \frac{(\sqrt{3} + 1)(\sqrt{3} + 1)}{3 - 1} \\ &= \frac{4 + 2\sqrt{3}}{2} = 2 + \sqrt{3}. \end{aligned}$$

$$\begin{aligned} \tan 15^\circ &= \tan(45^\circ - 30^\circ) = \frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \tan 30^\circ} \\ &= \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}} = \frac{\sqrt{3} - 1}{\sqrt{3} + 1} = \frac{(\sqrt{3} - 1)(\sqrt{3} - 1)}{(\sqrt{3} + 1)(\sqrt{3} - 1)} \\ &= \frac{4 - 2\sqrt{3}}{2} = 2 - \sqrt{3}. \end{aligned}$$

Ex. 2. Show that

$$(i) \tan(45^\circ + A) = \frac{1 + \tan A}{1 - \tan A}.$$

$$(ii) \tan(45^\circ - A) = \frac{1 - \tan A}{1 + \tan A}.$$

$$(i) \text{ Left side} = \frac{\tan 45^\circ + \tan A}{1 - \tan 45^\circ \tan A} = \frac{1 + \tan A}{1 - \tan A}.$$

(ii) This result follows similarly.

**Ex. 3.** Show that

$$\cot 2A + \tan A = \operatorname{cosec} 2A. \quad [C. U. 1947]$$

$$\begin{aligned} \text{Left side} &= \frac{\cos 2A}{\sin 2A} + \frac{\sin A}{\cos A} = \frac{\cos 2A \cos A + \sin 2A \sin A}{\sin 2A \cos A} \\ &= \frac{\cos (2A - A)}{\sin 2A \cos A} = \frac{\cos A}{\sin 2A \cos A} = \frac{1}{\sin 2A} \\ &= \operatorname{cosec} 2A. \end{aligned}$$

**39.** To find the expansions of

(i)  $\sin (A+B+C)$

(ii)  $\cos (A+B+C)$

(iii)  $\tan (A+B+C).$

(i)  $\sin (A+B+C)$

$$\begin{aligned} &= \sin \{(A+B)+C\} \\ &= \sin (A+B) \cos C + \cos (A+B) \sin C \\ &= (\sin A \cos B + \cos A \sin B) \cos C \\ &\quad + (\cos A \cos B - \sin A \sin B) \sin C \\ &= \sin A \cos B \cos C + \sin B \cos C \cos A \\ &\quad + \sin C \cos A \cos B - \sin A \sin B \sin C. \end{aligned}$$

**Note 1.** The expansion of  $\sin (A+B+C)$  can be easily put in the form

$$\cos A \cos B \cos C (\tan A + \tan B + \tan C - \tan A \tan B \tan C).$$

(ii)  $\cos (A+B+C)$

$$\begin{aligned} &= \cos \{(A+B)+C\} \\ &= \cos (A+B) \cos C - \sin (A+B) \sin C \\ &= (\cos A \cos B - \sin A \sin B) \cos C \\ &\quad - (\sin A \cos B + \cos A \sin B) \sin C \\ &= \cos A \cos B \cos C - \cos A \sin B \sin C \\ &\quad - \cos B \sin C \sin A - \cos C \sin A \sin B. \end{aligned}$$

**Note 2.** The expansion of  $\cos (A+B+C)$  can be easily put in the form

$$\cos A \cos B \cos C (1 - \tan B \tan C - \tan C \tan A - \tan A \tan B).$$

(iii)  $\tan (A+B+C)$ 

$$\begin{aligned}
 &= \tan \{(A+B)+C\} \\
 &= \frac{\tan (A+B)+\tan C}{1-\tan (A+B) \tan C} \\
 &= \frac{\frac{\tan A+\tan B}{1-\tan A \tan B}+\tan C}{1-\frac{\tan A+\tan B}{1-\tan A \tan B} \tan C} \\
 &= \frac{\tan A+\tan B+\tan C-\tan A \tan B \tan C}{1-\tan B \tan C-\tan C \tan A-\tan A \tan B}
 \end{aligned}$$

**Note 3** The expansion of  $\tan (A+B+C)$  can also be obtained thus,

$$\tan (A+B+C)=\frac{\sin (A+B+C)}{\cos (A+B+C)}.$$

Now, write down the expansions of  $\sin (A+B+C)$  and  $\cos (A+B+C)$  and divide the numerator and denominator by  $\cos A \cos B \cos C$  or simply write down the expansions of  $\sin (A+B+C)$  and  $\cos (A+B+C)$  as given in Notes 1 and 2.

**Obs.** Formulae for the Trigonometrical functions of the sum of four, five or more angles can be similarly obtained.

### Examples VI

Show that (Ex. 1 to 20) :—

1. (i)  $\sin (A-B)=\frac{3}{5}\frac{4}{5}$  and  $\cos (A+B)=\frac{3}{5}\frac{3}{5}$ ,

if  $A$  and  $B$  are acute and if  $\sin A=\frac{3}{5}$ ,  $\cos B=\frac{4}{5}$ .

(ii)  $\cos 68^{\circ} 20' \cos 8^{\circ} 20' + \cos 81^{\circ} 40' \cos 21^{\circ} 40' = \frac{1}{2}$ .

(iii)  $\sec (x-y)=\frac{5}{3}\frac{4}{3}$ , if  $\sec x=\frac{5}{3}$ ,  $\operatorname{cosec} y=\frac{4}{3}$ .

2. (i)  $\sin A \sin (B-C) + \sin B \sin (C-A)$   
 $+ \sin C \sin (A-B) = 0$ .

(ii)  $\cos A \sin (B-C) + \cos B \sin (C-A)$   
 $+ \cos C \sin (A-B) = 0$ .

(iii)  $\sin (B+C) \sin (B-C) + \sin (C+A) \sin (C-A)$   
 $+ \sin (A+B) \sin (A-B) = 0$ .

$$\begin{aligned} \text{(iv)} \quad & \sin(a-\theta) \sin(\beta-\gamma) + \sin(\beta-\theta) \sin(\gamma-a) \\ & + \sin(\gamma-\theta) \sin(a-\beta) = 0. \end{aligned}$$

$$\begin{aligned} 3. \quad & \cos(60^\circ - A) \cos(30^\circ - B) - \sin(60^\circ - A) \sin(30^\circ - B) \\ & = \sin(A+B). \end{aligned}$$

$$\begin{aligned} 4. \text{ (i)} \quad & \sin(n+1)x \cos(n-1)x - \cos(n+1)x \sin(n-1)x \\ & = \sin 2x. \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad & \sin 2\theta \cos \theta + \cos 2\theta \sin \theta \\ & = \sin 4\theta \cos \theta - \cos 4\theta \sin \theta. \end{aligned}$$

$$5. \quad \frac{\sin B}{\sin A} = \frac{\sin(2A+B)}{\sin A} - 2 \cos(A+B).$$

$$6. \quad \frac{\sin(B-C)}{\cos B \cos C} + \frac{\sin(C-A)}{\cos C \cos A} + \frac{\sin(A-B)}{\cos A \cos B} = 0.$$

$$7. \quad \frac{\sin(B-C)}{\sin B \sin C} + \frac{\sin(C-A)}{\sin C \sin A} + \frac{\sin(A-B)}{\sin A \sin B} = 0.$$

$$8. \quad \tan(A+B) \tan(A-B) = \frac{\sin^2 A - \sin^2 B}{\cos^2 A - \sin^2 B}.$$

$$9. \quad \tan^2 A - \tan^2 B = \frac{\sin(A+B) \sin(A-B)}{\cos^2 A \cos^2 B}.$$

$$10. \text{ (i)} \quad \frac{\tan(a+\beta) - \tan a}{1 + \tan(a+\beta) \tan a} = \tan \beta.$$

$$\begin{aligned} \text{(ii)} \quad & \text{If } A+B+C=\pi \text{ and } \cos A = \cos B \cos C, \text{ show that} \\ \tan A &= \tan B + \tan C. \quad [C. U. 1942] \end{aligned}$$

$$11. \quad 1 + \tan 2\theta \tan \theta = \sec 2\theta.$$

$$12. \quad \cot \theta - \cot 2\theta = \operatorname{cosec} 2\theta.$$

$$13. \quad \tan 20^\circ + \tan 25^\circ + \tan 25^\circ \tan 20^\circ = 1.$$

$$14. \text{ (i)} \quad \tan(45^\circ + A) = \frac{\cos A + \sin A}{\cos A - \sin A}.$$

$$\text{(ii)} \quad \sqrt{2} \sin(45^\circ + A) = \sin A + \cos A.$$

$$15. \quad \frac{\cos 8^\circ + \sin 8^\circ}{\cos 8^\circ - \sin 8^\circ} = \tan 53^\circ.$$

$$16. \tan (45^\circ + A) \tan (45^\circ - A) = 1.$$

$$17. \tan (A + B) + \tan (A - B) = \frac{\sin 2A}{\cos^2 A - \sin^2 B}.$$

$$18. \frac{\sin (x + y)}{\sin (x - y)} = \frac{\tan x + \tan y}{\tan x - \tan y}.$$

$$19. \cot (45^\circ + x) = \frac{\cot x - 1}{\cot x + 1} = \frac{\cos x - \sin x}{\cos x + \sin x}.$$

$$20. \sec (x + y) = \frac{\sec x \sec y}{1 - \tan x \tan y}.$$

$$21. \text{ Find the expansions of } \sin (A - B + C) \text{ and } \tan (A - B - C).$$

$$22. \text{ Express } \cot (A + B + C) \text{ in terms of } \cot A, \cot B, \cot C.$$

$$23. (i) \text{ If } a \cos (x + a) = b \cos (x - a), \text{ prove that } (a + b) \tan x = (a - b) \cot a.$$

$$(ii) \text{ If } \sin a \sin \beta - \cos a \cos \beta + 1 = 0, \text{ show that } 1 + \cot a \tan \beta = 0. \quad [C. U. 1939]$$

$$(iii) \text{ If } A + B + C = \pi \text{ and } \cos A = \cos B \cos C, \text{ then } \cot B \cot C = \frac{1}{2}.$$

$$24. \text{ If } \tan \theta = \frac{a \sin x + b \sin y}{a \cos x + b \cos y}, \text{ then } a \sin (\theta - x) + b \sin (\theta - y) = 0.$$

$$25. \text{ An angle } \theta \text{ is divided into two parts } \alpha, \beta \text{ such that } \tan \alpha : \tan \beta = x : y; \text{ prove that}$$

$$\sin (\alpha - \beta) = \frac{x - y}{x + y} \sin \theta.$$

$$26. \text{ If } \cos (\beta - \gamma) + \cos (\gamma - \alpha) + \cos (\alpha - \beta) = -\frac{3}{2}, \text{ show that } \Sigma \cos \alpha = 0 \text{ and } \Sigma \sin \alpha = 0.$$

## CHAPTER VII

### TRANSFORMATION OF PRODUCTS AND SUMS

#### 40. Transformation of products into sums or differences.

We have from Arts. 33 and 34,

$$\sin A \cos B + \cos A \sin B = \sin (A + B) \quad \dots \quad (1)$$

$$\sin A \cos B - \cos A \sin B = \sin (A - B). \quad \dots \quad (2)$$

Adding (1) and (2), we get

$$2 \sin A \cos B = \sin (A + B) + \sin (A - B). \quad \dots \quad (3)$$

Subtracting (2) from (1), we get

$$2 \cos A \sin B = \sin (A + B) - \sin (A - B). \quad \dots \quad (4)$$

Again, from Arts. 33 and 34, we have,

$$\cos A \cos B - \sin A \sin B = \cos (A + B) \quad \dots \quad (5)$$

$$\cos A \cos B + \sin A \sin B = \cos (A - B). \quad \dots \quad (6)$$

Adding (5) and (6), we get

$$2 \cos A \cos B = \cos (A + B) + \cos (A - B). \quad \dots \quad (7)$$

Subtracting (5) from (6), we get

$$2 \sin A \sin B = \cos (A - B) - \cos (A + B). \quad \dots \quad (8)$$

Thus, we have the following formulae for transforming a *product* of two sines and cosines into the *sum* or the *difference* of two sines or two cosines.

$$2 \sin A \cos B = \sin (A + B) + \sin (A - B). \quad \dots \quad (I)$$

$$2 \cos A \sin B = \sin (A + B) - \sin (A - B). \quad \dots \quad (II)$$

$$2 \cos A \cos B = \cos (A + B) + \cos (A - B). \quad \dots \quad (III)$$

$$2 \sin A \sin B = \cos (A - B) - \cos (A + B). \quad \dots \quad (IV)$$



#### 41. Transformation of sums or differences into products.

Let  $A + B = C$ , and  $A - B = D$ ,

$$\text{then } A = \frac{C+D}{2} \text{ and } B = \frac{C-D}{2}.$$

Making these substitutions for  $A$  and  $B$  in the results (3), (4), (7), (8) of Art. 40 and noting that the relation (8) can be written as

$$\begin{aligned} \cos(A+B) - \cos(A-B) &= -2 \sin A \sin B \\ &= -2 \sin A \sin(-B), \end{aligned}$$

we have the following four formulæ for transforming the *sum* or the *difference* of two sines only or two cosines only into a product of sines and cosines.

$$\sin C + \sin D = 2 \sin \frac{C+D}{2} \cos \frac{C-D}{2}. \quad \dots \quad \text{(I)}$$

$$\sin C - \sin D = 2 \cos \frac{C+D}{2} \sin \frac{C-D}{2}. \quad \dots \quad \text{(II)}$$

$$\cos C + \cos D = 2 \cos \frac{C+D}{2} \cos \frac{C-D}{2}. \quad \dots \quad \text{(III)}$$

$$\cos C - \cos D = 2 \sin \frac{C+D}{2} \sin \frac{D-C}{2}. \quad \dots \quad \text{(IV)}$$

**Note.** The following concise verbal statement of the above four formulæ is sometimes very convenient.

(i)  $\sin + \sin = 2 \sin (\frac{1}{2} \text{ sum}). \cos (\frac{1}{2} \text{ diff.}),$

(ii)  $\sin - \sin = 2 \cos (\frac{1}{2} \text{ sum}). \sin (\frac{1}{2} \text{ diff.}),$

(iii)  $\cos + \cos = 2 \cos (\frac{1}{2} \text{ sum}). \cos (\frac{1}{2} \text{ diff.}),$

(iv)  $\cos - \cos = 2 \sin (\frac{1}{2} \text{ sum}). \sin (\frac{1}{2} \text{ diff. reversed}).$

#### 42. Ex. 1. Prove that

$$(i) \cos 20^\circ \cos 40^\circ \cos 80^\circ = \frac{1}{8}.$$

$$(ii) \cos 80^\circ + \cos 40^\circ - \cos 20^\circ = 0.$$

$$\begin{aligned}
 \text{(i) Left side} &= \frac{1}{2} \cos 20^\circ (2 \cos 40^\circ \cos 80^\circ) \\
 &= \frac{1}{2} \cos 20^\circ (\cos 120^\circ + \cos 40^\circ) \\
 &= \frac{1}{2} \cos 20^\circ \left(-\frac{1}{2} + \cos 40^\circ\right) \\
 &= -\frac{1}{4} \cos 20^\circ + \frac{1}{2} \cos 20^\circ \cos 40^\circ \\
 &= -\frac{1}{4} \cos 20^\circ + \frac{1}{4} (\cos 60^\circ + \cos 20^\circ) \\
 &= -\frac{1}{4} \cos 20^\circ + \frac{1}{4} \left(\frac{1}{2} + \cos 20^\circ\right) \\
 &= \frac{1}{8}.
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii) Left side} &= (\cos 80^\circ + \cos 40^\circ) - \cos 20^\circ \\
 &= 2 \cos 60^\circ \cos 20^\circ - \cos 20^\circ \\
 &= 2 \cdot \frac{1}{2} \cos 20^\circ - \cos 20^\circ = 0.
 \end{aligned}$$

**Ex. 2.** Show that

$$\frac{\sin \theta + \sin 2\theta + \sin 4\theta + \sin 5\theta}{\cos \theta + \cos 2\theta + \cos 4\theta + \cos 5\theta} = \tan 3\theta.$$

$$\begin{aligned}
 \text{Numerator} &= (\sin 5\theta + \sin \theta) + (\sin 4\theta + \sin 2\theta) \\
 &= 2 \sin 3\theta \cos 2\theta + 2 \sin 3\theta \cos \theta \\
 &= 2 \sin 3\theta (\cos 2\theta + \cos \theta);
 \end{aligned}$$

$$\begin{aligned}
 \text{Denominator} &= (\cos 5\theta + \cos \theta) + (\cos 4\theta + \cos 2\theta) \\
 &= 2 \cos 3\theta \cos 2\theta + 2 \cos 3\theta \cos \theta \\
 &= 2 \cos 3\theta (\cos 2\theta + \cos \theta).
 \end{aligned}$$

$$\therefore \text{left side} = \frac{2 \sin 3\theta (\cos 2\theta + \cos \theta)}{2 \cos 3\theta (\cos 2\theta + \cos \theta)} = \frac{\sin 3\theta}{\cos 3\theta} = \tan 3\theta.$$

**Ex. 3.** Express  $4 \cos A \cos B \cos C$  as the sum of four cosines.

$$\begin{aligned}
 4 \cos A \cos B \cos C &= 2 \cos A (2 \cos B \cos C) \\
 &= 2 \cos A \{\cos (B+C) + \cos (B-C)\} \\
 &= 2 \cos A \cos (B+C) + 2 \cos A \cos (B-C) \\
 &= \cos (A+B+C) + \cos (A-B-C) \\
 &\quad + \cos (A+B-C) + \cos (A-B+C).
 \end{aligned}$$

**Ex. 4.** Express as the product of three sines

$$\begin{aligned}
 &\sin (B+C-A) + \sin (C+A-B) + \sin (A+B-C) \\
 &\quad - \sin (A+B+C).
 \end{aligned}$$

Grouping together the first two terms and grouping together the last two terms, the given expression

$$\begin{aligned}
 &= 2 \sin C \cos (B - A) + 2 \cos (A + B) \sin (-C) \\
 &= 2 \sin C \{\cos (B - A) - \cos (A + B)\} \\
 &= 2 \sin C (2 \sin B \sin A) \\
 &= 4 \sin A \sin B \sin C.
 \end{aligned}$$

### Examples VII

Prove that (Ex. 1 to 17) :—

1.  $\frac{\sin A + \sin B}{\sin A - \sin B} = \tan \frac{A+B}{2} \cot \frac{A-B}{2}.$
2.  $\frac{\cos A + \cos B}{\cos B - \cos A} = \cot \frac{A+B}{2} \cot \frac{A-B}{2}.$
3.  $\cos 20^\circ + \cos 100^\circ + \cos 140^\circ = 0.$
4.  $\sin \theta \sin (60^\circ - \theta) \sin (60^\circ + \theta) = \frac{1}{4} \sin 3\theta.$
5.  $\cos \theta \cos (60^\circ - \theta) \cos (60^\circ + \theta) = \frac{1}{4} \cos 3\theta.$
6.  $(\sin 3a + \sin a) \sin a + (\cos 3a - \cos a) \cos a = 0.$
7.  $\cos (A - D) \sin (B - C) + \cos (B - D) \sin (C - A) \\ + \cos (C - D) \sin (A - B) = 0.$
8.  $\cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ = \frac{1}{16}.$
9.  $\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ = \frac{1}{16}.$
10.  $\frac{\sin A + \sin B}{\cos A + \cos B} = \tan \frac{A+B}{2}.$
11.  $\frac{\sin A - \sin B}{\cos B - \cos A} = \cot \frac{A+B}{2}.$
12.  $\frac{\sin \theta + \sin 3\theta + \sin 5\theta + \sin 7\theta}{\cos \theta + \cos 3\theta + \cos 5\theta + \cos 7\theta} = \tan 4\theta.$
13.  $\frac{\sin 2A + \sin 5A - \sin A}{\cos 2A + \cos 5A + \cos A} = \tan 2A.$

$$14. \frac{\sin(\alpha + \beta) - 2 \sin \alpha + \sin(\alpha - \beta)}{\cos(\alpha + \beta) - 2 \cos \alpha + \cos(\alpha - \beta)} = \tan \alpha.$$

$$15. \frac{\cos 7\alpha + \cos 3\alpha - \cos 5\alpha - \cos \alpha}{\sin 7\alpha - \sin 3\alpha - \sin 5\alpha + \sin \alpha} = \cot 2\alpha.$$

$$16. \sin 2A + \sin 2B + \sin 2C - \sin 2(A + B + C) \\ = 4 \sin(B + C) \sin(C + A) \sin(A + B).$$

$$17. \cos A + \cos B + \cos C + \cos(A + B + C) \\ = 4 \cos \frac{B+C}{2} \cos \frac{C+A}{2} \cos \frac{A+B}{2}.$$

18. If  $\sin x = k \sin y$ , prove that

$$\tan \frac{1}{2}(x - y) = \frac{k - 1}{k + 1} \tan \frac{1}{2}(x + y).$$

19. If  $\cos x + \cos y = \frac{1}{2}$  and  $\sin x + \sin y = \frac{1}{4}$ , prove that  $\tan \frac{1}{2}(x + y) = \frac{3}{4}$ .

20. If  $x \cos \alpha + y \sin \alpha = k = x \cos \beta + y \sin \beta$ , prove that

$$\frac{x}{\cos \frac{1}{2}(\alpha + \beta)} = \frac{y}{\sin \frac{1}{2}(\alpha + \beta)} = \cos \frac{1}{2}(\alpha - \beta).$$

21. If  $\sin \theta + \sin \phi = a$ ,  $\cos \theta + \cos \phi = b$ , prove that

$$\tan \frac{\theta - \phi}{2} = \pm \sqrt{\frac{4 - a^2 - b^2}{a^2 + b^2}}.$$

22. Prove that  $\frac{\cos 10^\circ - \sin 10^\circ}{\cos 10^\circ + \sin 10^\circ} = \tan 35^\circ$ .

[Note that  $\sin \theta = \cos(90^\circ - \theta)$  and  $\cos \theta = \sin(90^\circ \pm \theta)$ .]

23. If  $\operatorname{cosec} A + \sec A = \operatorname{cosec} B + \sec B$ , then  $\tan A \tan B = \cot \frac{1}{2}(A + B)$  [P. U. 1936]

24. Prove that

$$\left( \frac{\cos A + \cos B}{\sin A - \sin B} \right)^n + \left( \frac{\sin A + \sin B}{\cos A - \cos B} \right)^n = 2 \cot^n \frac{A - B}{2},$$

or zero, according as  $n$  is even or odd. [P. U. 1938]

# CHAPTER VIII

## MULTIPLE ANGLES

### 43. Trigonometrical ratios of angle 2A.

From Art. 33, we have,

$$\begin{aligned}\sin (A+B) &= \sin A \cos B + \cos A \sin B, \\ \cos (A+B) &= \cos A \cos B - \sin A \sin B.\end{aligned}$$

Putting  $B=A$ , in the first formula, we get

$$\begin{aligned}\sin 2A &= \sin A \cos A + \cos A \sin A \\ &= 2 \sin A \cos A.\end{aligned}\quad \dots (1)$$

Putting  $B=A$ , in the second formula, we get

$$\begin{aligned}\cos 2A &= \cos A \cos A - \sin A \sin A \\ &= \cos^2 A - \sin^2 A\end{aligned}\quad \dots (2)$$

$$= (1 - \sin^2 A) - \sin^2 A = 1 - 2 \sin^2 A \quad \dots (3)$$

$$\text{and also } = \cos^2 A - (1 - \cos^2 A) = 2 \cos^2 A - 1. \quad \dots (4)$$

$$\text{By Art. 36, } \tan (A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

Putting  $B=A$ , in the above formula, we get

$$\tan 2A = \frac{\tan A + \tan A}{1 - \tan A \tan A} = \frac{2 \tan A}{1 - \tan^2 A}. \quad \dots (5)$$

Similarly, putting  $B=A$  in the value of  $\cot (A+B)$  as given in Art. 37, we get  $\cot 2A = \frac{\cot^2 A - 1}{2 \cot A}. \quad \dots (6)$

From formulæ (3) and (4), we obtain, by transposition,

$$1 + \cos 2A = 2 \cos^2 A \quad \dots \quad \dots (7)$$

$$1 - \cos 2A = 2 \sin^2 A. \quad \dots \quad \dots (8)$$

$$\text{By division, } \frac{1 - \cos 2A}{1 + \cos 2A} = \tan^2 A. \quad \dots \quad \dots (9)$$

We may also note that

$$\begin{aligned} 1 + \sin 2A &= \cos^2 A + \sin^2 A + 2 \sin A \cos A \\ &= (\cos A + \sin A)^2 \end{aligned}$$

$$\begin{aligned} 1 - \sin 2A &= \cos^2 A + \sin^2 A - 2 \sin A \cos A \\ &= (\cos A - \sin A)^2. \end{aligned}$$

**Note.** Since the addition formulæ are perfectly general (i.e., true for all values of  $A$  and  $B$ ), the above formulæ, being deduced from addition formulæ, are also perfectly general.

#### 44. Trigonometrical ratios of angle $3A$ .

$$\begin{aligned} \sin 3A &= \sin (A + 2A) = \sin A \cos 2A + \cos A \sin 2A \\ &= \sin A (1 - 2 \sin^2 A) + \cos A \cdot 2 \sin A \cos A \\ &\quad \quad \quad [ \text{By Art. 43} ] \\ &= \sin A (1 - 2 \sin^2 A) + 2 \sin A (1 - \sin^2 A). \end{aligned}$$

$$\therefore \sin 3A = 3 \sin A - 4 \sin^3 A.$$

$$\begin{aligned} \cos 3A &= \cos (A + 2A) = \cos A \cos 2A - \sin A \sin 2A \\ &= \cos A (2 \cos^2 A - 1) - \sin A \cdot 2 \sin A \cos A \\ &= \cos A (2 \cos^2 A - 1) - 2 \cos A \cdot \sin^2 A \\ &= 2 \cos^3 A - \cos A - 2 \cos A (1 - \cos^2 A). \end{aligned}$$

$$\therefore \cos 3A = 4 \cos^3 A - 3 \cos A.$$

$$\tan 3A = \tan (A + 2A) = \frac{\tan A + \tan 2A}{1 - \tan A \tan 2A}$$

$$\begin{aligned} &= \frac{\tan A + \frac{2 \tan A}{1 - \tan^2 A}}{1 - \tan A \cdot \frac{2 \tan A}{1 - \tan^2 A}} \\ &= \frac{\tan A (1 - \tan^2 A) + 2 \tan A}{(1 - \tan^2 A) - 2 \tan^2 A} \end{aligned}$$

$$\tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$$

**Obs.** By a method similar to that of the previous article the trigonometrical ratios of any higher multiple of  $A$  can be expressed in terms of those of  $A$ .

**45. Ex. 1.** Express  $\sin 2A$  and  $\cos 2A$  in terms of  $\tan A$ .

$$\sin 2A = 2 \sin A \cos A = 2 \frac{\sin A}{\cos A} \cdot \cos^2 A$$

$$= 2 \tan A \cdot \frac{1}{\sec^2 A} = \frac{2 \tan A}{1 + \tan^2 A}.$$

$$\cos 2A = \cos^2 A - \sin^2 A = \cos^2 A - \cos^2 A \cdot \frac{\sin^2 A}{\cos^2 A}$$

$$= \cos^2 A \left( 1 - \frac{\sin^2 A}{\cos^2 A} \right) = \frac{1}{\sec^2 A} (1 - \tan^2 A)$$

$$= \frac{1 - \tan^2 A}{1 + \tan^2 A}.$$

**Ex. 2.** Express  $\cos 4A$  in terms of  $\cos A$ .

Putting  $\theta = 2A$ ,  $\cos 4A = \cos 2\theta = 2 \cos^2 \theta - 1$

$$= 2 (\cos 2A)^2 - 1$$

$$= 2 (2 \cos^2 A - 1)^2 - 1$$

$$= 8 \cos^4 A - 8 \cos^2 A + 1.$$

**Ex. 3.** Show that  $\frac{1 - \tan^2 (45^\circ - A)}{1 + \tan^2 (45^\circ - A)} = \sin 2A$ .

Let  $\theta = 45^\circ - A$ ; then

$$\text{Left side} = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} = \frac{1 - \frac{\sin^2 \theta}{\cos^2 \theta}}{1 + \frac{\sin^2 \theta}{\cos^2 \theta}} = \frac{\cos^2 \theta - \sin^2 \theta}{\cos^2 \theta + \sin^2 \theta}$$

$$= \cos^2 \theta - \sin^2 \theta = \cos 2\theta$$

$$= \cos (90^\circ - 2A) = \sin 2A.$$

### Examples VIII

Prove the following identities (*Ex. 1 to 24*) :—

1.  $\frac{\sin 2A}{1 + \cos 2A} = \tan A.$
2.  $\frac{\sin 2A}{1 - \cos 2A} = \cot A.$
3.  $\cot A - \tan A = 2 \cot 2A.$
4. (i)  $(2 \cos \theta + 1)(2 \cos \theta - 1) = 2 \cos 2\theta + 1.$   
 (ii)  $\tan \theta (1 + \sec 2\theta) = \tan 2\theta.$
5.  $\frac{\cot A - \tan A}{\cot A + \tan A} = \cos 2A.$
6.  $\tan A + \cot A = 2 \operatorname{cosec} 2A.$
7.  $\cos^4 \theta - \sin^4 \theta = \cos 2\theta.$
8.  $\cos^6 \theta - \sin^6 \theta = \cos 2\theta (1 - \frac{1}{4} \sin^2 2\theta).$
9.  $\cos^6 \theta + \sin^6 \theta = \frac{1}{4} (1 + 3 \cos^2 2\theta).$
10.  $\frac{\sin^2 \alpha - \sin^2 \beta}{\sin \alpha \cos \alpha - \sin \beta \cos \beta} = \tan (\alpha + \beta).$
11. (i)  $\frac{1 - \cos 2\theta + \sin 2\theta}{1 + \cos 2\theta + \sin 2\theta} = \tan \theta. \quad [C. U. 1938]$   
 (ii)  $\frac{\sin \alpha - \sqrt{1 + \sin 2\alpha}}{\cos \alpha - \sqrt{1 + \sin 2\alpha}} = \cot \alpha. \quad [a \text{ being positive and acute, and the square root being taken with positive sign.}]$
12.  $\frac{\cos \theta + \sin \theta}{\cos \theta - \sin \theta} - \frac{\cos \theta - \sin \theta}{\cos \theta + \sin \theta} = 2 \tan 2\theta.$
13. (i)  $\frac{\sin A + \sin 2A}{1 + \cos A + \cos 2A} = \tan A.$   
 (ii)  $\frac{\sin 4\theta}{\cos 2\theta} \cdot \frac{1 - \cos 2\theta}{1 - \cos 4\theta} = \tan \theta.$
14. (i)  $\frac{\cos A - \sin A}{\cos A + \sin A} = \sec 2A - \tan 2A.$   
 (ii)  $\frac{\cos^3 \theta + \sin^3 \theta}{\cos \theta + \sin \theta} = 1 - \frac{1}{2} \sin 2\theta.$



$$15. \cos^3 A \cos 3A + \sin^3 A \sin 3A = \cos^3 2A.$$

$$16. (i) 4 (\cos^3 10^\circ + \sin^3 20^\circ) = 3 (\cos 10^\circ + \sin 20^\circ).$$

$$(ii) \cos 10^\circ - \sqrt{3} \sin 10^\circ = 2 \sin 20^\circ.$$

$$17. \tan 3\theta - \tan 2\theta - \tan \theta = \tan 3\theta \tan 2\theta \tan \theta.$$

$$18. \frac{\cot A}{\cot A - \cot 3A} - \frac{\tan A}{\tan 3A - \tan A} = 1.$$

$$19. \frac{1}{\tan 3\theta - \tan \theta} - \frac{1}{\cot 3\theta - \cot \theta} = \cot 2\theta.$$

$$20. \sin 8\theta = 8 \sin \theta \cos \theta \cos 2\theta \cos 4\theta.$$

$$21. (i) \cos 5\theta = 16 \cos^5 \theta - 20 \cos^3 \theta + 5 \cos \theta.$$

$$(ii) \sin 5\theta = 16 \sin^5 \theta - 20 \sin^3 \theta + 5 \sin \theta.$$

$$22. (i) \cot 3\theta = \frac{\cot^3 \theta - 3 \cot \theta}{3 \cot^2 \theta - 1}.$$

$$(ii) \tan 4\theta = \frac{4 \tan \theta - 4 \tan^3 \theta}{1 - 6 \tan^2 \theta + \tan^4 \theta}.$$

$$23. (i) \cos (120^\circ - A) + \cos A + \cos (120^\circ + A) = 0.$$

$$(ii) \cos^2 (A - 120^\circ) + \cos^2 A + \cos^2 (A + 120^\circ) = \frac{3}{2}.$$

$$24. \frac{\tan 2^n \theta}{\tan \theta} = (1 + \sec 2\theta)(1 + \sec 2^2 \theta) \cdots (1 + \sec 2^n \theta).$$

$$25. \text{ If } \alpha = \frac{\pi}{7}, \text{ show that}$$

$$(i) \cos \alpha \cos 2\alpha \cos 4\alpha = \frac{1}{8}.$$

$$(ii) \cos \alpha + \cos 2\alpha + \cos 4\alpha = -\frac{1}{2}.$$

$$(iii) \sin \alpha + \sin 2\alpha + \sin 4\alpha = \frac{\sqrt{7}}{8}.$$

$$26. \text{ If } \theta = \frac{\pi}{2^n + 1}, \text{ prove that}$$

$$2^n \cos \theta \cos 2\theta \cos 2^2 \theta \cdots \cos 2^{n-1} \theta = 1.$$

27. (i) If  $\tan x = b/a$ , find the value of  $a \cos 2x + b \sin 2x$ .

(ii) If  $\tan^2 x + 2 \tan x \tan 2y = \tan^2 y + 2 \tan y \tan 2x$ , prove that each side = 1, or, else,  $\tan x = \pm \tan y$ .

28. If  $\tan^2 \theta = 1 + 2 \tan^2 \phi$ , show that  $\cos 2\phi = 1 + 2 \cos 2\theta$ .

29. (i) If  $2 \tan \alpha = 3 \tan \beta$ , prove that

$$\tan(\alpha - \beta) = \frac{\sin 2\beta}{5 - \cos 2\beta} \quad [C. U. 1946]$$

(ii) If  $\frac{\tan(\alpha - \beta - \gamma)}{\tan(\alpha - \beta + \gamma)} = \frac{\tan \gamma}{\tan \beta}$ , show that

$$\text{either, } \sin(\beta - \gamma) = 0, \text{ or, } \sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0.$$

30. If  $\alpha$  and  $\beta$  are acute angles and  $\cos 2\alpha = \frac{3 \cos 2\beta - 1}{3 - \cos 2\beta}$ , show that  $\tan \alpha = \sqrt{2} \tan \beta$ . [C. U. 1941]

31. If  $\cos \theta = \frac{1}{2}(a + a^{-1})$ , show that

$$(i) \cos 2\theta = \frac{1}{2}(a^2 + a^{-2}).$$

$$(ii) \cos 3\theta = \frac{1}{2}(a^3 + a^{-3}).$$

Show that (Ex. 32 to 36):—

$$32. \sin^4 \theta = \frac{3}{8} - \frac{1}{2} \cos 2\theta + \frac{1}{8} \cos 4\theta.$$

$$33. \cos^3 \theta + \sin^3 \theta = 1 - \sin^2 2\theta + \frac{1}{8} \sin^4 2\theta.$$

$$34. \tan\left(\frac{\pi}{4} + A\right) + \tan\left(\frac{\pi}{4} - A\right) = 2 \sec 2A.$$

$$35. \cos^3 \theta \frac{\sin 3\theta}{3} + \sin^3 \theta \frac{\cos 3\theta}{3} = \frac{\sin 4\theta}{4}$$

$$36. \cos 4x - \cos 4y$$

$$= 8 (\cos x - \cos y)(\cos x + \cos y)(\cos x - \sin y) \\ \times (\cos x + \sin y).$$

## CHAPTER IX

### SUBMULTIPLE ANGLES

46. From the usual formulæ for multiple angles, namely

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2 \cos^2 A - 1 = 1 - 2 \sin^2 A$$

$$1 + \cos 2A = 2 \cos^2 A ; 1 - \cos 2A = 2 \sin^2 A$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

$$\sin 3A = 3 \sin A - 4 \sin^3 A$$

$$\cos 3A = 4 \cos^3 A - 3 \cos A$$

$$\tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$$

putting  $A = \frac{1}{2}\theta$  and  $\frac{1}{3}\theta$  respectively we derive the following formulæ for submultiple angles :

$$\sin \theta = 2 \sin \frac{1}{2}\theta \cos \frac{1}{2}\theta$$

$$\cos \theta = \cos^2 \frac{1}{2}\theta - \sin^2 \frac{1}{2}\theta = 2 \cos^2 \frac{1}{2}\theta - 1 = 1 - 2 \sin^2 \frac{1}{2}\theta$$

$$1 + \cos \theta = 2 \cos^2 \frac{1}{2}\theta ; 1 - \cos \theta = 2 \sin^2 \frac{1}{2}\theta$$

$$\tan \theta = \frac{2 \tan \frac{1}{2}\theta}{1 - \tan^2 \frac{1}{2}\theta}$$

$$\sin \theta = 3 \sin \frac{1}{3}\theta - 4 \sin^3 \frac{1}{3}\theta$$

$$\cos \theta = 4 \cos^3 \frac{1}{3}\theta - 3 \cos \frac{1}{3}\theta$$

$$\tan \theta = \frac{3 \tan \frac{1}{3}\theta - \tan^3 \frac{1}{3}\theta}{1 - 3 \tan^2 \frac{1}{3}\theta}$$

47. Values of  $\sin \frac{1}{2}\theta$  and  $\cos \frac{1}{2}\theta$  in terms of  $\cos \theta$ .

From  $\cos \theta = 2 \cos^2 \frac{1}{2}\theta - 1 = 1 - 2 \sin^2 \frac{1}{2}\theta$ , we at once deduce

$$\sin \frac{1}{2}\theta = \pm \sqrt{\frac{1}{2}(1 - \cos \theta)}$$

$$\cos \frac{1}{2}\theta = \pm \sqrt{\frac{1}{2}(1 + \cos \theta)}.$$

#### 48. Ambiguity of signs explained.

When  $\cos \theta$  is given and not  $\theta$ ,  $\theta$  and consequently  $\frac{1}{2}\theta$  has a series of values as will be explained in Chapter XI. Thus,  $\frac{1}{2}\theta$  may lie in any quadrant and  $\sin \frac{1}{2}\theta$  and  $\cos \frac{1}{2}\theta$  will then have corresponding signs.

If the quadrant in which  $\frac{1}{2}\theta$  lies be known, for example, when  $\theta$  is given along with  $\cos \theta$ , there is no ambiguity in choosing the proper signs of  $\cos \frac{1}{2}\theta$  and  $\sin \frac{1}{2}\theta$ , as shown in the following example.

**Ex.** Find  $\sin 22\frac{1}{2}^\circ$  and  $\cos 22\frac{1}{2}^\circ$ .

$$\sin 22\frac{1}{2}^\circ = + \sqrt{\frac{1}{2}(1 - \cos 45^\circ)} = \sqrt{\frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right)} = \frac{1}{2} \sqrt{2 - \sqrt{2}}$$

$$\cos 22\frac{1}{2}^\circ = + \sqrt{\frac{1}{2}(1 + \cos 45^\circ)} = \sqrt{\frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right)} = \frac{1}{2} \sqrt{2 + \sqrt{2}}.$$

#### 49. Values of $\sin \frac{1}{2}\theta$ and $\cos \frac{1}{2}\theta$ in terms of $\sin \theta$ .

We know that  $\sin \theta = 2 \sin \frac{1}{2}\theta \cos \frac{1}{2}\theta$

$$\text{and} \quad 1 = \cos^2 \frac{1}{2}\theta + \sin^2 \frac{1}{2}\theta.$$

Therefore,  $1 + \sin \theta = (\cos \frac{1}{2}\theta + \sin \frac{1}{2}\theta)^2$ ,

$$\text{and} \quad 1 - \sin \theta = (\cos \frac{1}{2}\theta - \sin \frac{1}{2}\theta)^2.$$

Hence,  $\cos \frac{1}{2}\theta + \sin \frac{1}{2}\theta = \pm \sqrt{1 + \sin \theta}$

$$\cos \frac{1}{2}\theta - \sin \frac{1}{2}\theta = \pm \sqrt{1 - \sin \theta}.$$

Thus,  $\cos \frac{1}{2}\theta = \pm \frac{1}{2} \sqrt{1 + \sin \theta} \pm \frac{1}{2} \sqrt{1 - \sin \theta}$

$$\text{and} \quad \sin \frac{1}{2}\theta = \pm \frac{1}{2} \sqrt{1 + \sin \theta} \mp \frac{1}{2} \sqrt{1 - \sin \theta}.$$

#### 50. Ambiguity of signs explained.

As before, when  $\sin \theta$  is given, and not  $\theta$ ,  $\theta$  has a series of values for the given value of  $\sin \theta$  as will be explained in Chapter XI;  $\frac{1}{2}\theta$  may therefore lie in any one of two possible quadrants.

$$\cos \frac{1}{2}\theta + \sin \frac{1}{2}\theta = \sqrt{2} \sin \left(\frac{1}{4}\pi + \frac{1}{2}\theta\right)$$

$$\text{and } \cos \frac{1}{2}\theta - \sin \frac{1}{2}\theta = \sqrt{2} \sin \left(\frac{1}{4}\pi - \frac{1}{2}\theta\right)$$

will have their signs determined accordingly.

Thus,  $\sin \frac{1}{2}\theta$  and  $\cos \frac{1}{2}\theta$  will be definitely known.

**Ex.** Find  $\sin 15^\circ$  and  $\cos 15^\circ$ .

We have,  $\cos 15^\circ + \sin 15^\circ = + \sqrt{1 + \sin 30^\circ} = \sqrt{1 + \frac{1}{2}}$

$$\cos 15^\circ - \sin 15^\circ = + \sqrt{1 - \sin 30^\circ} = \sqrt{1 - \frac{1}{2}}.$$

[  $\cos 15^\circ - \sin 15^\circ = \sqrt{2} \sin \left(\frac{1}{4}\pi - 15^\circ\right)$  and is clearly positive. ]

$$\text{Thus, } \cos 15^\circ = \frac{1}{2} \left( \sqrt{\frac{3}{2}} + \sqrt{\frac{1}{2}} \right) = \frac{\sqrt{3} + 1}{2\sqrt{2}}$$

$$\sin 15^\circ = \frac{1}{2} \left( \sqrt{\frac{3}{2}} - \sqrt{\frac{1}{2}} \right) = \frac{\sqrt{3} - 1}{2\sqrt{2}}.$$

### 51. $\tan \frac{1}{2}\theta$ in terms of $\tan \theta$ .

$$\text{From the formula, } \tan \theta = \frac{2 \tan \frac{1}{2}\theta}{1 - \tan^2 \frac{1}{2}\theta},$$

$$\text{i e., } \tan \theta \tan^2 \frac{1}{2}\theta + 2 \tan \frac{1}{2}\theta - \tan \theta = 0,$$

we easily deduce

$$\tan \frac{1}{2}\theta = \frac{-1 + \sqrt{1 + \tan^2 \theta}}{\tan \theta}.$$

The reason of the ambiguity is similar to those of the previous cases.

### 52. Ratios of $\frac{1}{3}\theta$ from those of $\theta$ .

By solving the cubic equation

$$\sin \theta = 3 \sin \frac{1}{3}\theta - 4 \sin^3 \frac{1}{3}\theta \quad \dots \quad \dots \quad (1)$$

we get  $\sin \frac{1}{3}\theta$ , if  $\sin \theta$  be known.

Similarly, by solving the cubic equations

$$\cos \theta = 4 \cos^3 \frac{1}{3}\theta - 3 \cos \frac{1}{3}\theta \quad \dots \quad \dots \quad (2)$$

$$\text{and } \tan \theta = \frac{3 \tan \frac{1}{3}\theta - \tan^3 \frac{1}{3}\theta}{1 - 3 \tan^2 \frac{1}{3}\theta} \quad \dots (3)$$

we derive values of  $\cos \frac{1}{3}\theta$  from those of  $\cos \theta$ , and of  $\tan \frac{1}{3}\theta$  from those of  $\tan \theta$  respectively.

### 53. Ratios of $18^\circ$ and $36^\circ$ .

Let  $\theta = 18^\circ$ ; then  $5\theta = 90^\circ$ ;  $\therefore 2\theta = 90^\circ - 3\theta$ .

$$\therefore \sin 2\theta = \cos 3\theta, \text{ or, } 2 \sin \theta \cos \theta = \cos \theta (4 \cos^2 \theta - 3).$$

As  $\cos \theta$  (i.e.  $\cos 18^\circ$ ) is not zero, we have

$$2 \sin \theta = 4 \cos^2 \theta - 3 = 1 - 4 \sin^2 \theta,$$

$$\text{or, } 4 \sin^3 \theta + 2 \sin \theta - 1 = 0.$$

$$\therefore \sin \theta = \frac{-2 \pm \sqrt{4+16}}{8} = \frac{(\pm \sqrt{5}-1)}{4}.$$

Now, as  $\theta$  here is a positive acute angle, therefore, rejecting the negative value, we get

$$\sin 18^\circ = \frac{1}{4}(\sqrt{5}-1).$$

$$\cos 18^\circ = + \sqrt{1 - \sin^2 18^\circ} = \frac{1}{4}(\sqrt{10+2\sqrt{5}}).$$

$$\cos 36^\circ = 1 - 2 \sin^2 18^\circ = \frac{1}{4}(\sqrt{5}+1).$$

$$\sin 36^\circ = \sqrt{1 - \cos^2 36^\circ} = \frac{1}{4}(\sqrt{10-2\sqrt{5}}).$$

**Note.** Since  $54^\circ$  and  $36^\circ$  are complementary and  $72^\circ$  and  $18^\circ$  are complementary, from the above values we easily get the trigonometrical ratios of  $54^\circ$  and  $72^\circ$ .

### 54. Ratios of $3^\circ$ and multiples of $3^\circ$ .

$$\begin{aligned} \sin 3^\circ &= \sin (18^\circ - 15^\circ) = \sin 18^\circ \cos 15^\circ - \cos 18^\circ \sin 15^\circ \\ &= \frac{1}{4}(\sqrt{5}-1)(\sqrt{6}+\sqrt{2}) - \frac{1}{4}(\sqrt{3}-1)(\sqrt{5}+\sqrt{5}), \end{aligned}$$

on substituting the values of  $\sin 18^\circ$ ,  $\cos 15^\circ$ , etc.

Similarly,

$$\cos 3^\circ = \frac{1}{4}(\sqrt{3}+1)(\sqrt{5}+\sqrt{5}) + \frac{1}{4}(\sqrt{6}-\sqrt{2})(\sqrt{5}-1).$$

From a knowledge of the ratios of  $3^\circ$ ,  $15^\circ$ ,  $18^\circ$ ,  $30^\circ$ ,  $36^\circ$  and  $45^\circ$ , we can deduce the ratios for all angles which are multiples of  $3^\circ$ , (for,  $6^\circ = 36^\circ - 30^\circ$ ;  $9^\circ = 45^\circ - 36^\circ$ ;

$12^\circ = 30^\circ - 18^\circ$ ;  $21^\circ = 36^\circ - 15^\circ$ ; etc.). For angles greater than  $45^\circ$ , the ratios may be deduced from those of their complements which are less than  $45^\circ$ .

Ex. Show that

$$\sin x = 2^n \cos \frac{x}{2} \cos \frac{x}{2^2} \cos \frac{x}{2^3} \cdots \cdots \cos \frac{x}{2^n} \sin \frac{x}{2^n}.$$

We have,  $\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2}$

$$\sin \frac{x}{2} = 2 \sin \frac{x}{2^2} \cos \frac{x}{2^2}$$

$$\sin \frac{x}{2^2} = 2 \sin \frac{x}{2^3} \cos \frac{x}{2^3}$$

$$\dots \quad \dots \quad \dots$$

Similarly,  $\sin \frac{x}{2^{n-1}} = 2 \sin \frac{x}{2^n} \cos \frac{x}{2^n}$ .

Hence,  $\sin x = 2^n \cos \frac{x}{2} \cos \frac{x}{2^2} \cos \frac{x}{2^3} \cdots \cdots \cos \frac{x}{2^n} \sin \frac{x}{2^n}$ .

### Examples IX

Prove that (Ex. 1 to 14):—

1.  $\frac{1 - \cos A}{\sin A} = \tan \frac{A}{2}$ .      2.  $\frac{1 + \cos A}{\sin A} = \cot \frac{A}{2}$ .

3.  $\left( \sin \frac{A}{2} \pm \cos \frac{A}{2} \right)^2 = 1 \pm \sin A$ .

4.  $\sec \theta + \tan \theta = \tan \left( \frac{1}{2}\pi + \frac{1}{2}\theta \right)$ . [ C. U. 1939 ]

5. (i)  $\frac{1 + \sin \theta - \cos \theta}{1 + \sin \theta + \cos \theta} = \tan \frac{\theta}{2}$ .

(ii)  $\frac{\sin \frac{1}{2}a - \sqrt{1 + \sin a}}{\cos \frac{1}{2}a - \sqrt{1 + \sin a}} = \cot \frac{a}{2}$  where  $0 < a < \pi$ ,

and the square root is taken with positive sign.

6. (i)  $\frac{1 + \sin x}{1 - \sin x} = \tan^2 \left( \frac{\pi}{2} + \frac{x}{2} \right)$ .

(ii)  $\frac{2 \sin \theta - \sin 2\theta}{2 \sin \theta + \sin 2\theta} = \tan^2 \frac{1}{2}\theta$ .

$$7. (i) \frac{1 + \tan \frac{1}{2}A}{1 - \tan \frac{1}{2}A} = \frac{1 + \sin A}{\cos A}.$$

$$(ii) \cot \beta = \frac{1}{2} (\cot \frac{1}{2}\beta - \tan \frac{1}{2}\beta).$$

$$(i) \frac{\sin 2\theta}{1 + \cos 2\theta} \cdot \frac{\cos \theta}{1 + \cos \theta} = \tan \frac{\theta}{2}.$$

$$(ii) 8 \sin^4 \frac{1}{2}\theta - 8 \sin^2 \frac{1}{2}\theta + 1 = \cos 2\theta.$$

$$9. \sin \theta = \frac{2 \tan \frac{1}{2}\theta}{1 + \tan^2 \frac{1}{2}\theta}.$$

$$10. \cos \theta = \frac{1 - \tan^2 \frac{1}{2}\theta}{1 + \tan^2 \frac{1}{2}\theta}.$$

$$11: (\cos x + \cos y)^2 + (\sin x + \sin y)^2 = 4 \cos^2 \frac{1}{2}(x - y).$$

$$12. \tan 6^\circ \tan 42^\circ \tan 66^\circ \tan 78^\circ = 1.$$

$$13. \tan 7\frac{1}{2}^\circ = \sqrt{6} - \sqrt{3} + \sqrt{2} - 2.$$

$$14. 2 \cos \frac{1}{18}\pi = \sqrt{2} + \sqrt{2} + \sqrt{2}.$$

$$15. (i) \text{ If } \tan \frac{\theta}{2} = \sqrt{\frac{1-e}{1+e}}, \tan \frac{\phi}{2}, \text{ show that}$$

$$\cos \phi = \frac{\cos \theta - e}{1 - e \cos \theta}.$$

(ii) If  $\tan \theta = \frac{\sin \alpha \sin \beta}{\cos \alpha + \cos \beta}$ , show that one of the values of  $\tan \frac{1}{2}\theta$  is  $\tan \frac{1}{2}\alpha \tan \frac{1}{2}\beta$ .

16. If  $\sin \alpha + \sin \beta = a$  and  $\cos \alpha + \cos \beta = b$ , find the value of  $\cos(\alpha + \beta)$ .

17. (i) Prove that  $2 \sin \frac{1}{2}A = \pm \sqrt{1 + \sin A} \pm \sqrt{1 - \sin A}$ , and determine which are the correct signs when  $270^\circ > A > 180^\circ$ . [B. H. U. I., 1931]

(ii) If  $\theta = 240^\circ$ , is the following statement correct?

$$2 \sin \frac{1}{2}\theta = \sqrt{1 + \sin \theta} - \sqrt{1 - \sin \theta}.$$

If not, how must it be modified?

18. If  $A = 320^\circ$ , prove that

$$\tan \frac{A}{2} = \frac{-1 + \sqrt{1 + \tan^2 A}}{\tan A}.$$



## CHAPTER X

### TRIGONOMETRICAL IDENTITIES

55. Many interesting identities involving functions of three or more angles can be established when there exists a relation among the angles. The most important of these identities are those in which the three angles are connected by the relation that their sum is equal to two right angles. In establishing this latter kind of identities, it will be necessary to make frequent use of the properties of supplementary and complementary angles.

Thus, since  $A + B + C = \pi$ ,

$$\therefore B + C = \pi - A.$$

$$\therefore \sin (B + C) = \sin (\pi - A) = \sin A.$$

Similarly,  $\sin (C + A) = \sin B$ ;  $\sin (A + B) = \sin C$ .

Again,  $\cos (B + C) = \cos (\pi - A) = -\cos A$ .

Similarly,  $\cos (C + A) = -\cos B$ ;  $\cos (A + B) = -\cos C$ .

$$\tan (B + C) = \tan (\pi - A) = -\tan A.$$

Similarly,  $\tan (C + A) = -\tan B$ ;  $\tan (A + B) = -\tan C$ .

Again, since,  $\frac{A}{2} + \frac{B}{2} + \frac{C}{2} = \frac{\pi}{2}$ ,

$$\therefore \sin \left( \frac{B}{2} + \frac{C}{2} \right) = \sin \left( \frac{\pi}{2} - \frac{A}{2} \right) = \cos \frac{A}{2}$$

Similarly,  $\sin \left( \frac{C}{2} + \frac{A}{2} \right) = \cos \frac{B}{2}$ ;

$$\sin \left( \frac{A}{2} + \frac{B}{2} \right) = \cos \frac{C}{2}.$$

Again,  $\cos \left( \frac{B}{2} + \frac{C}{2} \right) = \cos \left( \frac{\pi}{2} - \frac{A}{2} \right) = \sin \frac{A}{2}$

$$\text{Similarly, } \cos\left(\frac{C}{2} + \frac{A}{2}\right) = \sin \frac{B}{2};$$

$$\cos\left(\frac{A}{2} + \frac{B}{2}\right) = \sin \frac{C}{2}.$$

$$\tan\left(\frac{B}{2} + \frac{C}{2}\right) = \tan\left(\frac{\pi}{2} - \frac{A}{2}\right) = \cot \frac{A}{2}.$$

$$\text{Similarly, } \tan\left(\frac{C}{2} + \frac{A}{2}\right) = \cot \frac{B}{2};$$

$$\tan\left(\frac{A}{2} + \frac{B}{2}\right) = \cot \frac{C}{2}.$$

**56. Ex. 1.** *If  $A+B+C=\pi$ , prove that*

$$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C.$$

[*C. U. 1931, '33, '35; H. S. '61, Comp.*]

$$\text{Left side} = (\sin 2A + \sin 2B) + \sin 2C$$

$$= 2 \sin(A+B) \cos(A-B) + 2 \sin C \cos C$$

$$= 2 \sin C \cos(A-B) + 2 \sin C \cos C$$

[ $\because A+B+C=\pi$ .]

$$= 2 \sin C [\cos(A-B) + \cos C]$$

$$= 2 \sin C [\cos(A-B) - \cos(A+B)]$$

[ $\because A+B+C=\pi$ .]

$$= 2 \sin C \cdot 2 \sin A \sin B$$

$$= 4 \sin A \sin B \sin C.$$

**Ex. 2.** *If  $A+B+C=\pi$ , prove that*

$$\cos 2A + \cos 2B + \cos 2C = -4 \cos A \cos B \cos C - 1.$$

$$\text{Left side} = (\cos 2A + \cos 2B) + \cos 2C$$

$$= 2 \cos(A+B) \cos(A-B) + 2 \cos^2 C - 1$$

$$= -2 \cos C \cos(A-B) + 2 \cos^2 C - 1$$

[ $\because A+B+C=\pi$ .]

$$= -2 \cos C [\cos(A-B) - \cos C] - 1$$

$$= -2 \cos C [\cos(A-B) + \cos(A+B)] - 1$$

[ $\because A+B+C=\pi$ .]

$$= -2 \cos C \cdot 2 \cos A \cos B - 1$$

$$= -4 \cos A \cos B \cos C - 1.$$

**Ex. 3.** *If  $A+B+C=\pi$ , prove that*

$$\sin A + \sin B + \sin C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}.$$

[ *C. U. 1910, '29 ; H. S. '60, Comp.* ]

$$\text{Left side} = (\sin A + \sin B) + \sin C$$

$$= 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2} + 2 \sin \frac{C}{2} \cos \frac{C}{2}$$

$$= 2 \cos \frac{C}{2} \cos \frac{A-B}{2} + 2 \sin \frac{C}{2} \cos \frac{C}{2}$$

$$\left[ \because \frac{A}{2} + \frac{B}{2} + \frac{C}{2} = \frac{\pi}{2} \right]$$

$$= 2 \cos \frac{C}{2} \left[ \cos \frac{A-B}{2} + \sin \frac{C}{2} \right]$$

$$= 2 \cos \frac{C}{2} \left[ \cos \frac{A-B}{2} + \cos \frac{A+B}{2} \right]$$

$$\left[ \because \frac{A}{2} + \frac{B}{2} + \frac{C}{2} = \frac{\pi}{2} \right]$$

$$= 2 \cos \frac{C}{2} \cdot 2 \cos \frac{A}{2} \cos \frac{B}{2}$$

$$= 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}.$$

**Ex. 4.** *If  $A+B+C=\pi$ , prove that*

$$\cos A + \cos B + \cos C = 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}.$$

$$\text{Left side} = (\cos A + \cos B) + \cos C$$

$$= 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2} + 1 - 2 \sin^2 \frac{C}{2}$$

$$= 2 \sin \frac{C}{2} \cos \frac{A-B}{2} - 2 \sin^2 \frac{C}{2} + 1$$

$$\left[ \because \frac{A}{2} + \frac{B}{2} + \frac{C}{2} = \frac{\pi}{2} \right]$$

$$\begin{aligned}
&= 2 \sin \frac{C}{2} \left[ \cos \frac{A-B}{2} - \sin \frac{C}{2} \right] + 1 \\
&= 2 \sin \frac{C}{2} \left[ \cos \frac{A-B}{2} - \cos \frac{A+B}{2} \right] + 1 \\
&\quad \left[ \because \frac{A}{2} + \frac{B}{2} + \frac{C}{2} = \frac{\pi}{2} \right] \\
&= 2 \sin \frac{C}{2} \cdot 2 \sin \frac{A}{2} \sin \frac{B}{2} + 1 \\
&= 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}
\end{aligned}$$

**Ex. 5.** If  $A+B+C=\pi$ , prove that

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

[ H. S. 1961 ]

Since,  $B+C=\pi-A$ ,

$$\therefore \tan(B+C) = \tan(\pi-A).$$

$$\therefore \frac{\tan B + \tan C}{1 - \tan B \tan C} = -\tan A,$$

$$\begin{aligned}
\text{i.e., } \tan B + \tan C &= -\tan A (1 - \tan B \tan C) \\
&= -\tan A + \tan A \tan B \tan C.
\end{aligned}$$

$$\therefore \tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

Otherwise :

$$\tan(A+B+C) = \tan \pi = 0.$$

$$\therefore \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan B \tan C - \tan C \tan A - \tan A \tan B} = 0.$$

Since, the fraction is zero, numerator must be zero.

$$\therefore \tan A + \tan B + \tan C - \tan A \tan B \tan C = 0,$$

$$\text{i.e., } \tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

**Ex. 6.** If  $A+B+C=\pi$ , prove that

$$\tan \frac{B}{2} \tan \frac{C}{2} + \tan \frac{C}{2} \tan \frac{A}{2} + \tan \frac{A}{2} \tan \frac{B}{2} = 1.$$

[ C. U. 1936, '39 ]

Since,  $A + B + C = \pi$ ,  $\therefore \frac{A}{2} + \frac{B}{2} + \frac{C}{2} = \frac{\pi}{2}$ .

$$\therefore \tan\left(\frac{B}{2} + \frac{C}{2}\right) = \tan\left(\frac{\pi}{2} - \frac{A}{2}\right).$$

$$\therefore \frac{\tan \frac{B}{2} + \tan \frac{C}{2}}{1 - \tan \frac{B}{2} \tan \frac{C}{2}} = \cot \frac{A}{2} = \frac{1}{\tan \frac{A}{2}},$$

$$\text{or, } \tan \frac{A}{2} \left( \tan \frac{B}{2} + \tan \frac{C}{2} \right) = 1 - \tan \frac{B}{2} \tan \frac{C}{2}.$$

On simplification, the required result follows.

Otherwise :

$$\begin{aligned} 1 / \tan\left(\frac{A}{2} + \frac{B}{2} + \frac{C}{2}\right) &= \cot\left(\frac{A}{2} + \frac{B}{2} + \frac{C}{2}\right) = \cot \frac{\pi}{2} \\ \frac{1 - \tan \frac{B}{2} \tan \frac{C}{2} - \tan \frac{C}{2} \tan \frac{A}{2} - \tan \frac{A}{2} \tan \frac{B}{2}}{\tan \frac{A}{2} + \tan \frac{B}{2} + \tan \frac{C}{2} - \tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2}} &= 0. \end{aligned}$$

Now the value of the fraction being zero, its numerator must be zero.

$$\therefore 1 - \tan \frac{B}{2} \tan \frac{C}{2} - \tan \frac{C}{2} \tan \frac{A}{2} - \tan \frac{A}{2} \tan \frac{B}{2} = 0,$$

whence the required result follows.

**Ex. 7.** If  $A + B + C = \pi$ , prove that

$$\cos \frac{A}{2} + \cos \frac{B}{2} + \cos \frac{C}{2} = 4 \cos \frac{\pi - A}{4} \cos \frac{\pi - B}{4} \cos \frac{\pi - C}{4}.$$

$$\begin{aligned} \text{Right side} &= 2 \cos \frac{\pi - A}{4} \left[ 2 \cos \frac{\pi - B}{4} \cos \frac{\pi - C}{4} \right] \\ &= 2 \cos \frac{\pi - A}{4} \left[ \cos \frac{2\pi - (B + C)}{4} + \cos \frac{B - C}{4} \right] \\ &= 2 \cos \frac{\pi - A}{4} \left[ \cos \frac{\pi + A}{4} + \cos \frac{B - C}{4} \right] \end{aligned}$$

$$[\because 2\pi - (B + C) = \pi + \pi - (B + C) = \pi + A, \text{ since } A + B + C = \pi.]$$

$$\begin{aligned}
&= 2 \cos \frac{\pi-A}{4} \cos \frac{\pi+A}{4} + 2 \cos \frac{\pi-A}{4} \cos \frac{B-C}{4} \\
&= \left( \cos \frac{\pi}{2} + \cos \frac{A}{2} \right) + 2 \cos \frac{B+C}{4} \cos \frac{B-C}{4} \\
&\quad [\because A+B+C=\pi.] \\
&= \cos \frac{A}{2} + \cos \frac{B}{2} + \cos \frac{C}{2}.
\end{aligned}$$

**Note.** Since,  $\cos \frac{1}{2}(\pi-A) = \sin \left\{ \frac{1}{2}\pi - \frac{1}{2}(\pi-A) \right\} = \sin \frac{1}{2}(\pi+A)$   
and  $\cos \frac{1}{2}(\pi-A) = \cos \frac{1}{2}(A+B+C-A) = \cos \frac{1}{2}(B+C)$ .

$$\begin{aligned}
\therefore \text{ we have also, } &\cos \frac{1}{2}A + \cos \frac{1}{2}B + \cos \frac{1}{2}C \\
&= 4 \sin \frac{1}{4}(\pi+A) \sin \frac{1}{4}(\pi+B) \sin \frac{1}{4}(\pi+C) \\
&= 4 \cos \frac{1}{4}(B+C) \cos \frac{1}{4}(C+A) \cos \frac{1}{4}(A+B).
\end{aligned}$$

**Ex. 8.** If  $A+B+C=\pi$ , prove that

$$\cos^2 A + \cos^2 B + \cos^2 C + 2 \cos A \cos B \cos C = 1.$$

[C. U. 1932, '37, '47]

$$\begin{aligned}
&\cos^2 A + \cos^2 B + \cos^2 C \\
&= \frac{1}{2}(2 \cos^2 A + 2 \cos^2 B) + \cos^2 C \\
&= \frac{1}{2}(1 + \cos 2A + 1 + \cos 2B) + \cos^2 C \\
&= 1 + \frac{1}{2}(\cos 2A + \cos 2B) + \cos^2 C \\
&= 1 + \cos(A+B) \cos(A-B) + \cos C \cos C \\
&= 1 - \cos C \cos(A-B) - \cos C \cos(A+B) \\
&\quad [\because A+B=\pi-C.] \\
&= 1 - \cos C [\cos(A-B) + \cos(A+B)] \\
&= 1 - \cos C [2 \cos A \cos B] \\
&= 1 - 2 \cos A \cos B \cos C,
\end{aligned}$$

whence the required result follows.

**Ex. 9.** Show that

$$\begin{aligned}
&\tan(\beta-\gamma) + \tan(\gamma-\alpha) + \tan(\alpha-\beta) \\
&= \tan(\beta-\gamma) \tan(\gamma-\alpha) \tan(\alpha-\beta).
\end{aligned}$$

Let  $A = \beta - \gamma$ ,  $B = \gamma - \alpha$ ,  $C = \alpha - \beta$ ;

then  $A + B + C = \beta - \gamma + \gamma - \alpha + \alpha - \beta = 0$ .

$$\therefore \tan(A + B + C) = \tan 0 = 0.$$

$$\therefore \tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

Now, substituting the values for  $A, B, C$ , the required result follows.

**Ex. 10.** If  $x + y + z = xyz$ , prove that

$$x(1 - y^2)(1 - z^2) + y(1 - z^2)(1 - x^2) + z(1 - x^2)(1 - y^2) = 4xyz.$$

Putting  $x = \tan \alpha$ ,  $y = \tan \beta$ ,  $z = \tan \gamma$ , in the given relation, we have

$$\tan \alpha + \tan \beta + \tan \gamma = \tan \alpha \tan \beta \tan \gamma.$$

$\therefore$  by transposition,

$$\tan \alpha (1 - \tan \beta \tan \gamma) = -(\tan \beta + \tan \gamma).$$

$$\text{i.e., } \tan \alpha = -\frac{\tan \beta + \tan \gamma}{1 - \tan \beta \tan \gamma} = -\tan(\beta + \gamma).$$

$$\therefore \alpha = \pi - (\beta + \gamma). \therefore \alpha + \beta + \gamma = \pi. \therefore 2\alpha + 2\beta + 2\gamma = 2\pi.$$

$$\therefore \tan(2\alpha + 2\beta + 2\gamma) = \tan 2\pi = 0.$$

Therefore, as in Ex. 5 above,

$$\tan 2\alpha + \tan 2\beta + \tan 2\gamma = \tan 2\alpha \tan 2\beta \tan 2\gamma.$$

Now, expressing  $\tan 2\alpha$ ,  $\tan 2\beta$ ,  $\tan 2\gamma$  in terms of  $\tan \alpha$ ,  $\tan \beta$ ,  $\tan \gamma$  and substituting  $x, y, z$ , for them, we get,

$$\frac{2x}{1-x^2} + \frac{2y}{1-y^2} + \frac{2z}{1-z^2} = \frac{8xyz}{(1-x^2)(1-y^2)(1-z^2)}.$$

On simplification, the required result follows.

### Examples X

If  $A + B + C = \pi$ , prove that (Ex. 1 to 16) :—

$$\frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2}.$$

[ H. S. 1962 ]

$$2. \cot B \cot C + \cot C \cot A + \cot A \cot B = 1.$$

$$3. \cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}.$$

$$4. \tan 2A + \tan 2B + \tan 2C = \tan 2A \tan 2B \tan 2C.$$

$$5. (\cot B + \cot C)(\cot C + \cot A)(\cot A + \cot B) \\ = \operatorname{cosec} A \operatorname{cosec} B \operatorname{cosec} C.$$

$$6. \frac{\cot B + \cot C}{\tan B + \tan C} + \frac{\cot C + \cot A}{\tan C + \tan A} + \frac{\cot A + \cot B}{\tan A + \tan B} = 1.$$

$$7. \sin \frac{A}{2} + \sin \frac{B}{2} + \sin \frac{C}{2} \\ = 1 + 4 \sin \frac{\pi - A}{4} \sin \frac{\pi - B}{4} \sin \frac{\pi - C}{4} \\ = 1 + 4 \sin \frac{B + C}{4} \sin \frac{C + A}{4} \sin \frac{A + B}{4}.$$

$$8. \cos^2 2A + \cos^2 2B + \cos^2 2C \\ = 1 + 2 \cos 2A \cos 2B \cos 2C.$$

$$9. \sin^2 A + \sin^2 B + \sin^2 C = 2 + 2 \cos A \cos B \cos C.$$

$$10. \sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2} = 1 - 2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}.$$

$$11. \frac{\cos A}{\sin B \sin C} + \frac{\cos B}{\sin C \sin A} + \frac{\cos C}{\sin A \sin B} = 2.$$

[ C. U. 1949 ]

$$12. \frac{\sin 2A + \sin 2B + \sin 2C}{\sin A + \sin B + \sin C} = 8 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}.$$

$$13. \sin (B + C - A) + \sin (C + A - B) + \sin (A + B - C) \\ = 4 \sin A \sin B \sin C.$$

$$14. \sin (B + 2C) + \sin (C + 2A) + \sin (A + 2B) \\ = 4 \sin \frac{B - C}{2} \sin \frac{C - A}{2} \sin \frac{A - B}{2}.$$

$$15. \cos^2 A + \cos^2 B + 2 \cos A \cos B \cos C = \sin^2 C.$$

$$16. \cos \frac{A}{2} \cos \frac{B - C}{2} + \cos \frac{B}{2} \cos \frac{C - A}{2} + \cos \frac{C}{2} \cos \frac{A - B}{2} \\ = \sin A + \sin B + \sin C.$$



17. If  $\alpha + \beta + \gamma = \frac{1}{2}\pi$ , prove that

$$(i) \sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma + 2 \sin \alpha \sin \beta \sin \gamma = 1.$$

[ C. U. 1943 ]

$$(ii) \tan \beta \tan \gamma + \tan \gamma \tan \alpha + \tan \alpha \tan \beta = 1.$$

18. If  $A, B, C, D$  are the angles of a quadrilateral, prove that

$$(i) \frac{\tan A + \tan B + \tan C + \tan D}{\cot A + \cot B + \cot C + \cot D} = \tan A \tan B \tan C \tan D.$$

$$(ii) \cos A + \cos B + \cos C + \cos D = 4 \cos \frac{1}{2}(A+B) \cos \frac{1}{2}(B+C) \cos \frac{1}{2}(C+A).$$

19. Show that

$$(i) \cos^2(\beta - \gamma) + \cos^2(\gamma - \alpha) + \cos^2(\alpha - \beta) = 1 + 2 \cos(\beta - \gamma) \cos(\gamma - \alpha) \cos(\alpha - \beta).$$

$$(ii) \sin^2 \alpha + \sin^2 \beta + 2 \sin \alpha \sin \beta \cos(\alpha + \beta) = \sin^2(\alpha + \beta).$$

$$(iii) \cos^2 \theta + \cos^2(\alpha + \theta) - 2 \cos \alpha \cos \theta \cos(\alpha + \theta)$$

is independent of  $\theta$ .

20. (i) If  $\alpha + \beta = \gamma$ , show that

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1 + 2 \cos \alpha \cos \beta \cos \gamma.$$

[ C. U. 1940 ]

(ii) If  $\alpha + \beta + \gamma = 2\pi$ , show that

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma - 2 \cos \alpha \cos \beta \cos \gamma = 1.$$

21. If  $\cos(A+B) \sin(C+D) = \cos(A-B) \sin(C-D)$ , show that

$$\cot A \cot B \cot C = \cot D.$$

22. If  $A+B+C=2S$ , prove that

$$(i) \sin(S-A) + \sin(S-B) + \sin(S-C) - \sin S = 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}.$$

$$(ii) \cos^2 A + \cos^2 B + \cos^2 C + 2 \cos A \cos B \cos C - 1 = 4 \cos S \cos(S-A) \cos(S-B) \cos(S-C).$$

23. If  $A + B + C = n\pi$  ( $n$  being zero or an integer),  
then  $\tan A + \tan B + \tan C = \tan A \tan B \tan C$ .
24. Show that, if  $\alpha + \beta + \gamma = \pi$ ,  

$$\tan(\beta + \gamma - \alpha) + \tan(\gamma + \alpha - \beta) + \tan(\alpha + \beta - \gamma)$$

$$= \tan(\beta + \gamma - \alpha) \tan(\gamma + \alpha - \beta) \tan(\alpha + \beta - \gamma).$$
25. If  $A + B + C = \pi$ , prove that
- (i)  $\sin A \cos B \cos C + \sin B \cos C \cos A$   

$$+ \sin C \cos A \cos B = \sin A \sin B \sin C.$$
- (ii)  $\cos A \sin B \sin C + \cos B \sin C \sin A$   

$$+ \cos C \sin A \sin B = 1 + \cos A \cos B \cos C.$$
- (iii)  $\sin 5A + \sin 5B + \sin 5C$   

$$= 4 \cos \frac{5A}{2} \cos \frac{5B}{2} \cos \frac{5C}{2}.$$
- (iv)  $(\tan A + \tan B + \tan C)(\cot A + \cot B + \cot C)$   

$$= 1 + \sec A \sec B \sec C.$$
26. If  $\cos A + \cos B + \cos C = 0$ , show that  

$$\cos 3A + \cos 3B + \cos 3C = 12 \cos A \cos B \cos C.$$
  
 [ Write  $\cos 3A = 4 \cos^3 A - 3 \cos A$ , etc. ]
27. If  $x + y + z = \frac{1}{2}\pi$ , prove that  

$$\cos(x - y - z) + \cos(y - z - x) + \cos(z - x - y)$$

$$- 4 \cos x \cos y \cos z = 0.$$
28. Show that  

$$\sin(y - z) + \sin(z - x) + \sin(x - y)$$

$$+ 4 \sin \frac{y - z}{2} \sin \frac{z - x}{2} \sin \frac{x - y}{2} = 0.$$
29. If  $x + y + z = 0$ , show that  

$$\cot(z + x - y) \cot(x + y - z) + \cot(x + y - z) \cot(y + z - x)$$

$$+ \cot(y + z - x) \cot(z + x - y) = 1.$$
30. If  $x + y + z = xyz$ , prove that  

$$\frac{3x - x^3}{1 - 3x^2} + \frac{3y - y^3}{1 - 3y^2} + \frac{3z - z^3}{1 - 3z^2} = \frac{3x - x^3}{1 - 3x^2} \cdot \frac{3y - y^3}{1 - 3y^2} \cdot \frac{3z - z^3}{1 - 3z^2}.$$

## CHAPTER XI

### TRIGONOMETRICAL EQUATIONS AND GENERAL VALUES

**57.** It will be apparent from Chapter IV that there are infinitely many angles, the trigonometrical ratios of which have a given value. For example, if  $\sin \theta = \frac{1}{2}$ , one value of  $\theta$  (the smallest positive value) is known to be  $30^\circ$ . Now, sines of supplementary angles are equal. Hence,  $\sin 150^\circ$  being equal to  $\sin 30^\circ$  is also  $\frac{1}{2}$ . Again, angles differing from  $30^\circ$  or  $150^\circ$  by complete multiples of  $360^\circ$  will have their sines (in fact all ratios) the same. Thus, sine of each of the angles  $30^\circ, 150^\circ, 390^\circ, 510^\circ, -330^\circ, -210^\circ$ , etc. is equal to  $\frac{1}{2}$ . Similarly, if  $\cos \theta$  be given, equal to  $\frac{1}{\sqrt{2}}$  say,  $\theta$  may have any of the values  $+45^\circ, +315^\circ, +405^\circ, -315^\circ, -45^\circ$ , etc.; or else, if  $\tan \theta = \sqrt{3}$ ,  $\theta$  may have any of the values  $60^\circ, 240^\circ, 420^\circ, -300^\circ$ , etc.

It is very convenient for the solution of trigonometrical equations, as also for other purposes, to obtain a general expression in a compact form embracing all angles, the trigonometrical ratios of which have a given value.

**58.** General expression of all angles, one of whose trigonometrical ratios is zero.

, If the sine of an angle be zero, from definition, the length of the perpendicular from any point of one of its arms upon another is zero, so that the two arms must be in the same straight line. Evidently, therefore, such angles must be zero, or some multiple of  $\pi$ , odd or even.

Thus, if  $\sin \theta = 0$ , then  $\theta = n\pi$ ,  
*n being zero, or any integer, positive or negative.*

When the cosine of an angle is zero, the projection of any length along one arm upon another is zero, and so the two arms must be at right angles to one another. The angles must therefore be evidently either  $\frac{\pi}{2}$  or  $\frac{3\pi}{2}$  or differ from these by complete revolutions; in other words, the angle may be any odd multiple of  $\frac{\pi}{2}$ .

Thus, if  $\cos \theta = 0$ , then  $\theta = (2n+1)\frac{\pi}{2}$ ,

*n* being zero, or any integer, positive or negative.

Again, if  $\tan \theta = 0$ , then its numerator  $\sin \theta$  is also zero; and so  $\theta = n\pi$ .

Similarly, if  $\cot \theta = 0$ , then  $\cos \theta = 0$ ;

and so  $\theta = (2n+1)\frac{\pi}{2}$ .

**Note.** The ratios  $\operatorname{cosec} \theta$  or  $\sec \theta$  can never be zero, for they can never be numerically less than unity.

### 59. General expression of angles having the same sine (or cosecant).

Let  $a$  be any angle positive or negative such that its sine is equal to a given quantity  $k$  (numerically not greater than 1); for fixing up the idea, and for the sake of convenience in practice, the smallest positive angle having its sine for the given quantity  $k$  is taken as  $a$ . Let  $\theta$  be any other angle whose sine is equal to  $k$ .

Then,  $\sin \theta = \sin a$ ,

or,  $\sin \theta - \sin a = 0$ ,

or,  $2 \sin \frac{1}{2}(\theta - a) \cos \frac{1}{2}(\theta + a) = 0$ .

$\therefore$  either  $\sin \frac{1}{2}(\theta - a) = 0$ ,

i.e.,  $\frac{1}{2}(\theta - a) = \text{any multiple of } \pi = m\pi, \quad \dots (1)$

or, else  $\cos \frac{1}{2}(\theta + \alpha) = 0$ ,

i.e.,  $\frac{1}{2}(\theta + \alpha) = \text{any odd multiple of } \frac{\pi}{2} = (2m+1) \frac{\pi}{2} \dots (2)$

From (1),  $\theta - \alpha = 2m\pi$ , i.e.,  $\theta = \alpha + 2m\pi \dots (3)$

From (2),  $\theta + \alpha = (2m+1)\pi$ , i.e.,  $\theta = -\alpha + (2m+1)\pi \dots (4)$

Combining (3) and (4),  $\theta = (-1)^n \alpha + n\pi \dots (5)$

where  $n$  is zero, or any integer, positive or negative, odd or even.

If  $\operatorname{cosec} \theta = \operatorname{cosec} \alpha$ , then  $\sin \theta = \sin \alpha$ ; hence all angles having the same cosecant as that of  $\alpha$  are also given by the expression (5).

Thus, all angles having the same sine or cosecant as that of  $\alpha$  are given by  $2n\pi + \alpha$  and  $(2n+1)\pi - \alpha$ ,

or,  $n\pi + (-1)^n \alpha$ .

## 60. General expression of angles having the same cosine (or secant).

Let  $\alpha$  be the smallest positive angle such that its cosine is equal to a given quantity  $k$  (numerically  $\leq 1$ ); and let  $\theta$  be any other angle whose cosine is equal to  $k$ .

Then,  $\cos \theta = \cos \alpha$ ,

or,  $\cos \alpha - \cos \theta = 0$ ,

$\therefore 2 \sin \frac{1}{2}(\theta + \alpha) \sin \frac{1}{2}(\theta - \alpha) = 0$ .

$\therefore$  either  $\sin \frac{1}{2}(\theta + \alpha) = 0$ ,

i.e.,  $\frac{1}{2}(\theta + \alpha) = \text{any multiple of } \pi = n\pi \dots (1)$

or else,  $\sin \frac{1}{2}(\theta - \alpha) = 0$ ,

i.e.,  $\frac{1}{2}(\theta - \alpha) = \text{any multiple of } \pi = n\pi \dots (2)$

From (1),  $\theta + \alpha = 2n\pi$ , or,  $\theta = 2n\pi - \alpha \dots (3)$

From (2),  $\theta - \alpha = 2n\pi$ , or,  $\theta = 2n\pi + \alpha \dots (4)$

From (3) and (4), we have  $\theta = 2n\pi \pm \alpha$ ,  $\dots$  (5)  
 where  $n$  is zero, or any integer, positive or negative.

It is also evident as in the previous case that all angles having the same secant as that of  $\alpha$  are also included in the expression (5).

Hence, *all angles having the same cosine or secant as that of  $\alpha$  are given by*

$$2n\pi \pm \alpha,$$

*$n$  being zero, or any integer, positive or negative.*

**Note.** As in Art. 59, instead of taking the smallest positive angle, we might take  $\alpha$  to be any one angle having for its cosine the given quantity  $k$ . The general values of  $\theta$  satisfying  $\cos \theta = \cos \alpha$  as obtained above, would not be affected at all.

### 61. General expression of all angles having the same tangent (or cotangent).

Let  $\alpha$  be the smallest positive angle such that its tangent is equal to a given quantity  $k$ ; and let  $\theta$  be any other angle whose tangent is equal to  $k$ .

Then,  $\tan \theta = \tan \alpha$ ,

$$\text{or, } \frac{\sin \theta}{\cos \theta} - \frac{\sin \alpha}{\cos \alpha} = 0,$$

$$\text{or, } \frac{\sin \theta \cos \alpha - \cos \theta \sin \alpha}{\cos \theta \cos \alpha} = 0,$$

$$\text{or, } \frac{\sin (\theta - \alpha)}{\cos \theta \cos \alpha} = 0.$$

$$\therefore \sin (\theta - \alpha) = 0,$$

$$\text{i.e., } \theta - \alpha = \text{any multiple of } \pi = n\pi.$$

$$\therefore \theta = \alpha + n\pi. \quad \dots (1)$$

The factor  $\frac{1}{\cos \theta \cos \alpha}$  cannot be zero, for cosine of an angle cannot have an infinitely large value.

It is also evident as in the previous case that all angles having the same cotangent as that of  $\alpha$  are given by the expression (1).

Hence, *all angles having the same tangent or cotangent as that of  $\alpha$  are given by*

$$n\pi + \alpha,$$

*n being zero, or any integer, positive or negative.*

**Note.** The remark below Art. 60 is applicable here also.

## 62. Special cases.

From Art. 59, considering both cases when  $n$  is odd or even, it may be easily seen that

$$\text{if } \sin \theta = 1 = \sin \frac{\pi}{2}, \quad \theta = 2n\pi + \frac{\pi}{2} = (4n+1) \frac{\pi}{2};$$

$$\text{and if } \sin \theta = -1 = \sin \left( -\frac{\pi}{2} \right), \quad \theta = 2n\pi - \frac{\pi}{2} = (4n-1) \frac{\pi}{2}$$

$$\text{or,} \quad \quad \quad = (4k+3) \frac{\pi}{2},$$

*where  $n$  (or  $k=n-1$ ) is zero, or any integer, positive or negative.*

Similarly, from Art. 60, it may be seen that

$$\text{if } \cos \theta = 1, \quad \theta = 2n\pi$$

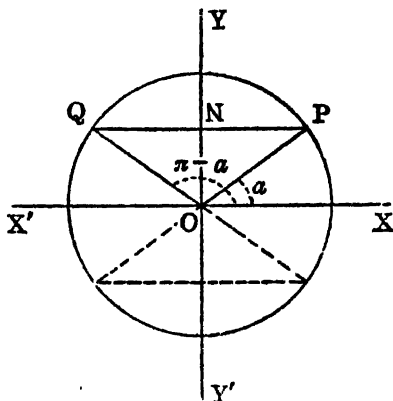
$$\text{and if } \cos \theta = -1, \quad \theta = (2n+1) \pi,$$

*n being zero, or any integer, positive or negative.*

These are the usual forms in which the above special cases are used in practice.

## 63. Geometrical Treatment.

(i) *Geometrical construction of an angle whose sine (or cosecant) is given, and to obtain a general expression of all such angles.*



Let the sine of an angle be given equal to ' $a$ '.

Taking the perpendicular lines  $XOX'$  and  $YOY'$  for reference, draw a circle of unit radius with centre  $O$ .

Measure off  $ON = a$  along  $OY$  (or along  $OY'$  if  $a$  be negative). Through  $N$  draw a straight line  $PNQ$  parallel to  $XOX'$  meeting the circle at  $P$  and  $Q$ .

Then,  $\angle POX = a$  say, is one of the required angles, for  
 $\sin a = \sin OPN = \frac{ON}{OP} = a$ .

Another angle with the same sine, as is apparent from the figure, is  $\angle QOX = \pi - a$  (or  $3\pi - a$  if  $a = ON$  be negative, which is trigonometrically the same as  $\pi - a$ ).

' $a$ ' being given in magnitude and sign, the position of  $N$  on  $YOY'$  is fixed and thus in one revolution, i.e. from 0 to

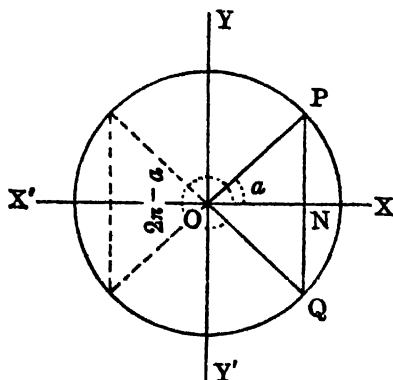


$2\pi$  there are, as is clear from the figure, only two angles  $a$  and  $\pi - a$  having the given sine.\*

Now, the addition or subtraction of any multiple of  $2\pi$  makes no difference in the values of the trigonometrical ratios of an angle (See Art. 28).

Hence, all the angles having the same sine as that of  $a$  are contained in the formulæ  $2m\pi + a$  and  $2m\pi + \pi - a$  i.e.,  $(2m + 1)\pi - a$ , where  $m$  is zero, or any integer, positive or negative. Both the sets of angles are evidently included in the formula  $n\pi + (-1)^n a$ ,  $n$  being zero, or any integer, positive or negative.

(ii) *Angles with given cosine ( or secant ).*



Let the given cosine be ' $a$ '. As before, measure off  $ON = a$  along  $OX$  (or along  $OX'$  if ' $a$ ' be negative), and through  $N$  draw  $PNQ$  parallel to  $YOY'$  to meet the circle with centre  $O$  and radius unity, at  $P$  and  $Q$ .

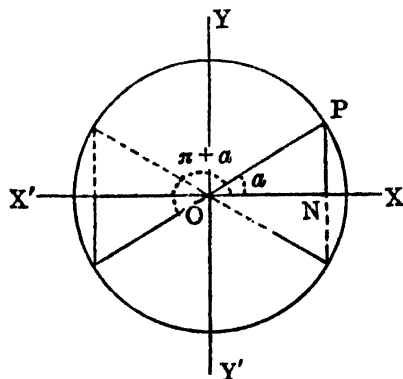
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\* In the same quadrant there cannot be two distinct angles (without being coterminals) having the same sine, for the corresponding triangles will then be congruent.

Let  $\angle POX = a$ . Then,  $a$  is a required angle. Also from the figure, the only angles in the first four quadrants which have the given cosine are  $a$  and  $2\pi - a$ .

Adding or subtracting multiples of  $2\pi$  to these, all the angles having the same cosine as that of  $a$  are given by  $2m\pi + a$  or  $2m\pi + 2\pi - a$ , both of which are included in the formula  $2n\pi \pm a$ ,  $n$  being zero, or any integer, positive or negative.

(iii) *Angles with given tangent (or cotangent).*



Let ' $a$ ' be the given tangent. Along  $OX$  or  $OX'$  measure off  $ON$  of unit length, and then measure off  $NP$  perpendicular to it of length whose numerical value is ' $a$ '. If ' $a$ ' be positive, both  $ON$  and  $NP$  will be positive, or both will be negative, and so the  $\angle XOP$  will be either in the first or in the third quadrant. If ' $a$ ' be negative, the angle will be either in the second or in the fourth quadrant. In any case there are only two angles, within one revolution, i.e., from 0 to  $2\pi$  as is apparent from the figure, with the given tangent.\*

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\*The ratio  $PN : ON$  being given, and the included angle  $PNO$  being right, the triangle  $PNO$  constructed remains always similar to itself and so in the same quadrant  $\angle PON$  of the triangle is unique.

One of the angles being  $\alpha$ , the other is evidently (from the figure)  $\pi + \alpha$ . Adding or subtracting multiples of  $2\pi$ , all the angles having the same tangent as that of  $\alpha$  are given by  $2m\pi + \alpha$  or  $2m\pi + \pi + \alpha$  both of which are included in the formula  $n\pi + \alpha$  where  $n$  is zero, or any integer, positive or negative, odd or even.

**Ex. 1.** Solve  $2(\cos^2\theta - \sin^2\theta) = 1$ .

The given equation can be written as

$$2 \cos 2\theta = 1. \quad \therefore \cos 2\theta = \frac{1}{2} = \cos \frac{1}{3}\pi.$$

$$\therefore 2\theta = 2n\pi \pm \frac{1}{3}\pi. \quad \therefore \theta = n\pi \pm \frac{1}{6}\pi.$$

**Note.** It may be observed that a trigonometrical equation can be solved in several ways; and the results though different in forms will give the same series of angles. To illustrate this we work out the above example in another way.

The equation can also be written in the form

$$2(\cos^2\theta - 1 + \cos^2\theta) = 1, \text{ or, } 4 \cos^2\theta = 3.$$

$$\therefore \cos \theta = \pm \frac{\sqrt{3}}{2} = \cos \frac{\pi}{6}, \text{ or, } \cos \frac{5\pi}{6}.$$

$$\therefore \theta = 2m\pi \pm \frac{\pi}{6}, \text{ or, } 2m\pi \pm \frac{5\pi}{6}.$$

$$\text{Now, } 2m\pi \pm \frac{5\pi}{6} = (2m+1)\pi - \frac{\pi}{6}, \text{ or, } (2m-1)\pi + \frac{\pi}{6}.$$

All the four sets of solutions,  $m$  being any integer, can be included in the expression  $n\pi \pm \frac{1}{6}\pi$ , in which form the result has already been obtained by the previous process.

**Ex. 2.** Solve  $4 \cos^2 x + 6 \sin^2 x = 5$ .

The equation can be written as

$$4 \cos^2 x + 6 \sin^2 x = 5 (\sin^2 x + \cos^2 x).$$

$$\therefore \sin^2 x = \cos^2 x, \text{ or, } \tan^2 x = 1.$$

$$\therefore \tan x = \pm 1. \quad \therefore x = n\pi \pm \frac{1}{4}\pi.$$

**Note.** Equations of the form  $a \cos^2 x + b \sin^2 x = c$  can be easily solved by the above method, or by expressing sine in terms of cosine or cosine in terms of sine.

**Ex. 3.** Solve  $2 \sin^2 x + \sin^2 2x = 2$ . [ C. U. 1940 ]

The given equation can be written as

$$2(1 - \sin^2 x) - \sin^2 2x = 0, \text{ or, } 2 \cos^2 x - 4 \sin^2 x \cos^2 x = 0.$$

$$\text{or, } 2 \cos^2 x (1 - 2 \sin^2 x) = 0, \quad \text{or, } \cos^2 x \cos 2x = 0.$$

$$\therefore \text{ either } \cos x = 0, \text{ i.e., } x = n\pi + \frac{1}{2}\pi,$$

$$\text{or, } \cos 2x = 0, \text{ i.e., } 2x = 2n\pi \pm \frac{1}{2}\pi. \quad \therefore x = n\pi \pm \frac{1}{4}\pi.$$

**Ex. 4.** Solve  $\cos \theta - \sin \theta = \frac{1}{\sqrt{2}}$ .

Dividing both sides of the equation by  $\sqrt{1^2 + 1^2}$ , i.e.,  $\sqrt{2}$ , we have

$$\cos \theta \cdot \frac{1}{\sqrt{2}} - \sin \theta \cdot \frac{1}{\sqrt{2}} = \frac{1}{2},$$

$$\text{i.e., } \cos \theta \cos \frac{1}{2}\pi - \sin \theta \sin \frac{1}{2}\pi = \frac{1}{2}.$$

$$\therefore \cos(\theta + \frac{1}{2}\pi) = \cos \frac{1}{2}\pi. \quad \therefore \theta + \frac{1}{2}\pi = 2n\pi \pm \frac{1}{2}\pi.$$

$$\therefore \theta = 2n\pi + \frac{1}{2}\pi, \text{ or, } 2n\pi - \frac{7}{2}\pi.$$

**Note.** Extraneous solutions.

In general, as pointed out in Ex. 1 above, the same trigonometrical equation may be solved by different methods, and the forms of the result we arrive at, though apparently different in some cases, are ultimately equivalent. In some cases, however, we may be tempted to solve a trigonometrical equation by methods which have flaws in them leading to solutions which include in addition to the correct solutions, some extraneous solutions which do not satisfy the given equation. The given equation which is of the type  $a \cos \theta + b \sin \theta = c$  is an example. We proceed to demonstrate it as follows:

$$\text{Here, } \cos \theta - \frac{1}{\sqrt{2}} = \sin \theta.$$

$$\therefore \cos^2 \theta - \sqrt{2} \cos \theta + \frac{1}{2} = \sin^2 \theta = 1 - \cos^2 \theta,$$

$$\text{whence } 2 \cos^2 \theta - \sqrt{2} \cos \theta - \frac{1}{2} = 0.$$

$$\therefore \cos \theta = \frac{\sqrt{2} \pm \sqrt{2 + 4}}{4} = \frac{1 \pm \sqrt{3}}{2\sqrt{2}} = \cos \frac{\pi}{12}, \quad \text{or, } \cos \frac{7\pi}{12}.$$

$$\therefore \theta = 2n\pi \pm \frac{1}{12}\pi, \quad \text{or, } 2n\pi \pm \frac{7}{12}\pi.$$

But it can be easily seen on substitution that

$2n\pi - \frac{1}{12}\pi$  and  $2n\pi + \frac{7}{12}\pi$  do not satisfy the given equation. *The error in the method lies in squaring the equation as we have done;* for the squared equation includes the equation  $\cos \theta - \frac{1}{\sqrt{2}} = -\sin \theta$ , i.e.,  $\cos \theta + \sin \theta = \frac{1}{\sqrt{2}}$  of which the solutions are  $2n\pi - \frac{1}{12}\pi$  and  $2n\pi + \frac{7}{12}\pi$ .

Equations of this type are therefore best solved as in the next example, and not by squaring.

Thus, while solving any trigonometrical equation it is always advisable to verify the roots obtained; for thereby extraneous roots, if any, can be easily detected.

**Ex. 5.** Solve  $a \cos \theta + b \sin \theta = c$ . ( $c > \sqrt{a^2 + b^2}$ )

Put  $a = r \cos \alpha$ ,  $b = r \sin \alpha$ , choosing the smallest positive value of  $\alpha$ , keeping  $r$  positive.

$$\text{Then, } r = \sqrt{a^2 + b^2} \text{ and } \sin \alpha = \frac{b}{\sqrt{a^2 + b^2}}$$

$$\text{and } \cos \alpha = \frac{a}{\sqrt{a^2 + b^2}}.$$

The signs of  $a$  and  $b$  will determine the quadrant in which  $\alpha$  lies, and  $a$  and  $b$  being given,  $r$  and  $\alpha$  are definitely known.

The equation now becomes

$$r \cos(\theta - \alpha) = c,$$

$$\text{or, } \cos(\theta - \alpha) = \frac{c}{\sqrt{a^2 + b^2}} = \cos \beta,$$

where  $\beta$  is the smallest positive angle whose cosine is  $\frac{c}{\sqrt{a^2 + b^2}}$  and  $a, b, c$  being known,  $\beta$  is also known.

$$\text{Hence, } \theta - \alpha = 2n\pi \pm \beta, \text{ or, } \theta = 2n\pi + \alpha \pm \beta.$$

**Note.** An angle which is introduced in a trigonometrical work to facilitate calculations is called a *subsidiary angle*. Thus,  $\alpha$  and  $\beta$  are here subsidiary angles.

**Ex. 6.** Solve  $4 \cos x + 5 \sin x = 5$ , given  $\tan 51^\circ 21' = \frac{5}{4}$ .

Dividing both sides of the given equation by  $\sqrt{4^2 + 5^2}$ , i.e., by  $\sqrt{41}$ , we get

$$\frac{4}{\sqrt{41}} \cos x + \frac{5}{\sqrt{41}} \sin x = \frac{5}{\sqrt{41}}. \quad \dots (1)$$

$$\text{Since, } \tan 51^\circ 21' = \frac{5}{4},$$

$$\therefore \sin 51^\circ 21' = \frac{5}{\sqrt{41}}, \cos 51^\circ 21' = \frac{4}{\sqrt{41}}.$$

$\therefore$  (1) reduces to

$$\cos x \cos 51^\circ 21' + \sin x \sin 51^\circ 21' = \sin 51^\circ 21'$$

$$\text{or, } \cos (x - 51^\circ 21') = \sin 51^\circ 21' = \cos 38^\circ 39'.$$

$$\therefore x - 51^\circ 21' = 2n\pi \pm 38^\circ 39'.$$

$$\therefore x = 2n\pi + 90^\circ, \text{ or, } 2n\pi + 12^\circ 42'.$$

**Ex. 7.** (i) Solve  $2 \sin^2 x + \sin^2 2x = 2$  for  $-\pi < x < \pi$ .

From Ex. 3 above, we see that  $x = n\pi + \frac{1}{2}\pi \dots$  (1)

$$\text{or, } x = n\pi \pm \frac{1}{2}\pi. \dots \dots \dots$$
 (2)

Putting  $n = 0, -1$  in (1), we get  $x = \frac{1}{2}\pi, -\frac{1}{2}\pi$ , which lie in the given interval. Putting  $n = 0, 1, -1$  in (2), we get  $x = \pm \frac{1}{2}\pi, \frac{3}{2}\pi, -\frac{3}{2}\pi$  which also lie in the given interval.

Hence, the required values of  $x$  are  $\pm \frac{1}{2}\pi, \pm \frac{1}{2}\pi, \pm \frac{3}{2}\pi$ .

(ii) Solve  $\cos \theta + \sqrt{3} \sin \theta = 2$

$$\text{for } -2\pi < \theta < 2\pi \text{ and } 3\pi < \theta < 5\pi.$$

Dividing both sides of the equation by  $\sqrt{1+3}$ , i.e., 2, we have

$$\cos \theta \cdot \frac{1}{2} + \sin \theta \cdot \frac{\sqrt{3}}{2} = 1,$$

$$\text{i.e., } \cos \theta \cdot \cos \frac{1}{2}\pi + \sin \theta \cdot \sin \frac{1}{2}\pi = 1,$$

$$\text{i.e., } \cos (\theta - \frac{1}{2}\pi) = 1.$$

$$\therefore \theta - \frac{1}{2}\pi = 2n\pi. \text{ i.e., } \theta = 2n\pi + \frac{1}{2}\pi.$$

Putting  $n = 0, -1$ , we get  $\theta = \frac{1}{2}\pi, -\frac{1}{2}\pi$  which lie in the 1st interval.

Again, putting  $n = 1, 2$ , we get  $\theta = \frac{5}{2}\pi, \frac{9}{2}\pi$ , which lie in the 2nd interval.

**Ex. 8.** Solve  $\tan ax = \cot bx$ .

Here,  $\tan ax = \cot bx = \tan (\frac{1}{2}\pi - bx)$ .

$$\therefore ax = n\pi + \frac{1}{2}\pi - bx.$$

$$\therefore x = \frac{2n+1}{a+b} \cdot \frac{\pi}{2}.$$

## Examples XI

Solve the following equations (*Ex. 1 to 23*) :—

1.  $\cot^2 x + \operatorname{cosec}^2 x = 3$ .

2. (i)  $2 \cos^2 \theta + 4 \sin^2 \theta = 3$ .

(ii)  $\tan^2 \theta = 3 \operatorname{cosec}^2 \theta - 1$ . [ C. U. 1939 ]

3.  $\tan x - \cot x = \operatorname{cosec} x$ .

4.  $\cot x - \cot 2x = 2$ .

5.  $2 \sin \theta \tan \theta + 1 = \tan \theta + 2 \sin \theta$ .

6.  $\sin 5\theta + \sin \theta = \sin 3\theta$ .

7.  $\sin m\theta + \sin n\theta = 0$ .

8.  $\cos x + \cos 3x + \cos 5x + \cos 7x = 0$ .

9.  $\cot 2x = \cos x + \sin x$ .

10.  $\sin x + \cos x = \sqrt{2}$ , for  $-\pi < x < \pi$ .

11.  $\sin 2x \tan x + 1 = \sin 2x + \tan x$ .

12.  $\cot x - \tan x = 2$ . [ C. U. 1934, '37 ]

13.  $\sin x + \sqrt{3} \cos x = \sqrt{2}$ . [ C. U. 1938, '47 ]

14.  $2 \sin x \sin 3x = 1$ .

15.  $\sin \theta + 2 \cos \theta = 1$ . [ C. U. 1933 ]

16.  $\tan x + \tan 2x + \tan 3x = \tan x \tan 2x \tan 3x$ .

17.  $\tan(\frac{1}{2}\pi + \theta) + \tan(\frac{1}{2}\pi - \theta) = 4$ . [ C. U. 1949 ]

18.  $\tan x + \tan 2x + \tan x \tan 2x = 1$ . [ C. U. 1941, '45 ]

19.  $\cos \theta + \sqrt{3} \sin \theta = \sqrt{2}$ . [ C. U. 1944 ]

20.  $\sqrt{3} \cos x + \sin x = 1$ , for  $-2\pi < x < 2\pi$ .

21.  $\cos 2x = \cos x \sin x$ .

22.  $2 \cot x + \sin x = 2 \operatorname{cosec} x$ .

23.  $\cos x + \sin x = \cos 2x + \sin 2x$ . [ C. U. 1943 ]

24. Solve  $2 \sin^2 x + \sin x = 3$ ; and find all the angles between  $0^\circ$  and  $1000^\circ$  which satisfy it.

25. Find the solution of the equations (general solution is not required)

$$\tan x + \tan y = 2$$

$$2 \cos x \cos y = 1.$$

26. If  $\tan ax - \tan bx = 0$ , show that the values of  $x$  form a series in A.P.

27. Solve

(i)  $\cos 3x + \cos 2x + \cos x = 0$ . [ C. U. 1941, '49 ]

(ii)  $\cos 9x \cos 7x = \cos 5x \cos 3x$ ,  $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$ .

(iii)  $\tan x + \tan 2x + \tan 3x = 0$ . [ A. I. 1941 ]

(iv)  $\cos x - \sin x = \cos a + \sin a$ . [ B. II. U. 1938 ]

(v)  $\cos^3 x - \cos x \sin x - \sin^3 x = 1$ .

(vi)  $\cos 6x + \cos 4x = \sin 3x + \sin x$ .

(vii)  $\frac{\sin a}{\sin 2x} + \frac{\cos a}{\cos 2x} = 2$ .

28. Solve  $5 \cos \theta + 2 \sin \theta = 2$ , given  $\tan 68^\circ 12' = 2\frac{1}{2}$ .

29. Find those pairs of solutions of the following equations which correspond to positive solutions less than  $2\pi$  of each individual equation :—

(i)  $\sin (a - \beta) = 0$  ;  $\sin (a + \beta) = 1$ .

(ii)  $\sin (a - \beta) = \cos (a + \beta) = \frac{1}{2}$ .

30. If  $\sin A = \sin B$ ,  $\cos A = \cos B$ , prove that either  $A$  and  $B$  are equal or they differ by some multiple of four right angles. [ C. U. 1935 ]

31. Show that the three equations

$$\sin^2 \theta = \sin^2 a, \cos^2 \theta = \cos^2 a, \tan^2 \theta = \tan^2 a$$

are all identical and the solution is always  $n\pi \pm a$ .

32. Show that the same two series of angles are given by the equations

$$x + \frac{\pi}{4} = n\pi + (-1)^n \frac{\pi}{6} \text{ and } x - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{3}.$$



## CHAPTER XII

### INVERSE CIRCULAR FUNCTIONS

64. The equation  $\sin \theta = x$  means that  $\theta$  is an angle whose sine is  $x$ . It is often convenient to express this statement *inversely* by writing  $\theta = \sin^{-1} x$ . Thus, the symbol  $\sin^{-1} x$  denotes an angle whose sine is  $x$ . Hence,  $\sin^{-1} x$  is an angle, whereas  $\sin \theta$  is a number. The two relations  $\sin \theta = x$  and  $\theta = \sin^{-1} x$  are identical; if one is given the other follows. The symbol  $\sin^{-1} x$  is usually read as "*sine inverse x*". Sometimes it is also denoted by *arc sin x*.

**Note.**  $\sin^{-1} x$  must not be confused with  $(\sin x)^{-1}$ , i.e.,  $\frac{1}{\sin x}$ .

65. We know that if  $\theta$  be any one angle whose sine is equal to  $x$ , then sines of all the angles given by  $n\pi + (-1)^n \theta$  are equal to  $x$ . Hence,  $\sin^{-1} x$  has got an infinite number of values, and as such,  $\sin^{-1} x$  is a *multiple-valued function*.

Hence, the *general value* of  $\sin^{-1} x = n\pi + (-1)^n \sin^{-1} x$  where on the right-hand side  $\sin^{-1} x$  stands for any particular angle whose sine is  $x$ .

Similarly, the *general value* of

$$\cos^{-1} x = 2n\pi \pm \cos^{-1} x$$

$$\text{and of } \tan^{-1} x = n\pi + \tan^{-1} x.$$

The smallest numerical value, either positive or negative, of  $\theta$  is called the *principal value* of  $\sin^{-1} x$ . Thus, the principal value of  $\sin^{-1} \frac{1}{2}$  is  $30^\circ$ . If corresponding to the same ratio, there are two numerically equal angles, one positive and the other negative, it is customary to take the positive angle as the principal value; thus, the principal value of  $\cos^{-1} \frac{1}{2}$  is  $60^\circ$ , and not  $(-60^\circ)$  although  $\cos(-60^\circ) = \frac{1}{2}$ .

In all numerical examples, the principal value is generally taken.

$\cos^{-1}x$ ,  $\tan^{-1}x$ ,  $\operatorname{cosec}^{-1}x$ ,  $\sec^{-1}x$ ,  $\cot^{-1}x$  have similar significance and all properties as those of  $\sin^{-1}x$ . These expressions are called **Inverse Circular Functions**.

66. If  $\sin \theta = x$ , then  $\theta = \sin^{-1}x$ , i.e.,  $\theta = \sin^{-1} \sin \theta$ .

Similarly,  $\theta = \cos^{-1} \cos \theta = \tan^{-1} \tan \theta$ ; etc.

Again, if  $\theta = \sin^{-1}x$ ,  $\sin \theta = x$ , i.e.,  $\sin \sin^{-1}x = x$ .

Similarly,  $\cos \cos^{-1}x = x$ ;  $\tan \tan^{-1}x = x$ ; etc.

Also, we have

$$\operatorname{cosec}^{-1}x = \sin^{-1} \frac{1}{x}; \cot^{-1}x = \tan^{-1} \frac{1}{x}; \sec^{-1}x = \cos^{-1} \frac{1}{x}.$$

Let  $\operatorname{cosec}^{-1}x = \theta$ ; then  $\operatorname{cosec} \theta = x$ .

$$\therefore \sin \theta = \frac{1}{\operatorname{cosec} \theta} = \frac{1}{x}.$$

Hence,  $\theta = \sin^{-1} \frac{1}{x}$ , and therefore,  $\operatorname{cosec}^{-1}x = \sin^{-1} \frac{1}{x}$ .

In the same way we have,  $\operatorname{cosec}^{-1} \frac{1}{x} = \sin^{-1}x$ .

The other relations follow similarly.

67. As all the trigonometrical ratios can be expressed in terms of any one, similarly all the inverse trigonometrical ratios can be expressed in terms of any one inverse ratio.

Thus, let  $\sin^{-1}x = \theta$ ; then  $\sin \theta = x$ ,

$$\therefore \cos \theta = \sqrt{1-x^2}; \tan \theta = \frac{x}{\sqrt{1-x^2}}; \cot \theta = \frac{\sqrt{1-x^2}}{x};$$

$$\sec \theta = \frac{1}{\cos \theta} \quad \text{and} \quad \operatorname{cosec} \theta = \frac{1}{\sin \theta}.$$

$$\begin{aligned} \therefore \theta &= \sin^{-1}x = \cos^{-1} \sqrt{1-x^2} = \tan^{-1} \frac{x}{\sqrt{1-x^2}} \\ &= \cot^{-1} \frac{\sqrt{1-x^2}}{x} = \sec^{-1} \frac{1}{\sqrt{1-x^2}} = \operatorname{cosec}^{-1} \frac{1}{x}. \end{aligned}$$

68. To prove that

$$(i) \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}.$$

$$(ii) \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}.$$

$$(iii) \operatorname{cosec}^{-1}x + \sec^{-1}x = \frac{\pi}{2}.$$

(i) Let  $\sin^{-1}x = \theta$  ; then  $\sin \theta = x$ .

Now,  $\sin \theta = \cos (\frac{1}{2}\pi - \theta)$ .

$\therefore \cos (\frac{1}{2}\pi - \theta) = x$  and hence  $\cos^{-1}x = \frac{1}{2}\pi - \theta$ .

Therefore,  $\sin^{-1}x + \cos^{-1}x = \theta + \frac{1}{2}\pi - \theta = \frac{1}{2}\pi$ .

(ii) Let  $\tan^{-1}x = \theta$  ; then  $\tan \theta = x$ .

Now,  $\tan \theta = \cot (\frac{1}{2}\pi - \theta)$ .

$\therefore \cot (\frac{1}{2}\pi - \theta) = x$ .  $\therefore \cot^{-1}x = \frac{1}{2}\pi - \theta$ .

$\therefore \tan^{-1}x + \cot^{-1}x = \theta + \frac{1}{2}\pi - \theta = \frac{1}{2}\pi$ .

(iii) Let  $\operatorname{cosec}^{-1}x = \theta$  ; then  $\operatorname{cosec} \theta = x$ .

Now,  $\operatorname{cosec} \theta = \sec (\frac{1}{2}\pi - \theta)$ .

$\therefore \sec (\frac{1}{2}\pi - \theta) = x$ .  $\therefore \sec^{-1}x = \frac{1}{2}\pi - \theta$ .

$\therefore \operatorname{cosec}^{-1}x + \sec^{-1}x = \theta + \frac{1}{2}\pi - \theta = \frac{1}{2}\pi$ .

69. To prove that

$$(i) \tan^{-1}x + \tan^{-1}y = \tan^{-1} \frac{x+y}{1-xy}.$$

$$(ii) \tan^{-1}x - \tan^{-1}y = \tan^{-1} \frac{x-y}{1+xy}.$$

Let  $\tan^{-1}x = \alpha$  ; and  $\tan^{-1}y = \beta$  ;

then  $\tan \alpha = x$  ; and  $\tan \beta = y$ .

Now,  $\tan (\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} = \frac{x+y}{1-xy}$ .

$$\therefore a + \beta = \tan^{-1} \frac{x+y}{1-xy},$$

$$\text{i.e., } \tan^{-1}x + \tan^{-1}y = \tan^{-1} \frac{x+y}{1-xy}.$$

$$\text{Again, } \tan(a - \beta) = \frac{\tan a - \tan \beta}{1 + \tan a \tan \beta} = \frac{x-y}{1+xy}.$$

$$\therefore a - \beta = \tan^{-1} \frac{x-y}{1+xy},$$

$$\text{i.e., } \tan^{-1}x - \tan^{-1}y = \tan^{-1} \frac{x-y}{1+xy}.$$

Note. It can be easily proved as above that

$$\cot^{-1}x \pm \cot^{-1}y = \cot^{-1} \frac{xy \mp 1}{y \pm x}.$$

70. To prove that

$$\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1} \frac{x+y+z-xyz}{1-yz-zx-xy}.$$

$$\text{Let } \tan^{-1}x = a; \tan^{-1}y = \beta; \tan^{-1}z = \gamma.$$

$$\therefore \tan a = x, \tan \beta = y, \tan \gamma = z.$$

Now,  $\tan(a + \beta + \gamma)$

$$\begin{aligned} &= \frac{\tan a + \tan \beta + \tan \gamma - \tan a \tan \beta \tan \gamma}{1 - \tan \beta \tan \gamma - \tan \gamma \tan a - \tan a \tan \beta} \\ &= \frac{x + y + z - xyz}{1 - yz - zx - xy}. \end{aligned}$$

$$\text{Hence, } a + \beta + \gamma = \tan^{-1} \frac{x+y+z-xyz}{1-yz-zx-xy}.$$

Since,  $a + \beta + \gamma = \tan^{-1}x + \tan^{-1}y + \tan^{-1}z$ , the required result follows.

Note. This relation can also be deduced by applying twice the formula of Art. 69. Thus,

$$\text{Left side} = (\tan^{-1}x + \tan^{-1}y) + \tan^{-1}z$$

$$= \tan^{-1} \frac{x+y}{1-xy} + \tan^{-1}z; \text{ now again apply Art. 69.}$$

71. In fact, for most of the formulæ involving ordinary circular functions, corresponding relations connecting the inverse circular functions can be easily deduced. In addition to those given above, some are illustrated in the following examples :

**Ex. 1.** Show that

$$(i) \sin^{-1}x \pm \sin^{-1}y = \sin^{-1}\{x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\}.$$

$$(ii) \cos^{-1}x \pm \cos^{-1}y = \cos^{-1}\{xy \mp \sqrt{(1-x^2)(1-y^2)}\}.$$

(i) Let  $\sin^{-1}x = \alpha$ .  $\therefore \sin \alpha = x$  and  $\cos \alpha = \sqrt{1-x^2}$  ;  
also let  $\sin^{-1}y = \beta$ .  $\therefore \sin \beta = y$  and  $\cos \beta = \sqrt{1-y^2}$ .

$$\begin{aligned} \text{Now, } \sin(\alpha \pm \beta) &= \sin \alpha \cos \beta \pm \cos \alpha \sin \beta \\ &= x\sqrt{1-y^2} \pm y\sqrt{1-x^2}. \end{aligned}$$

$$\therefore \alpha \pm \beta = \sin^{-1}\{x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\}.$$

Since,  $\alpha \pm \beta = \sin^{-1}x \pm \sin^{-1}y$ , the required result follows.

(ii) These relations follow similarly from the value of  $\cos(\alpha \pm \beta)$ .

**Ex. 2.** Show that

$$(i) 2 \sin^{-1}x = \sin^{-1}(2x\sqrt{1-x^2}).$$

$$(ii) 2 \cos^{-1}x = \cos^{-1}(2x^2 - 1).$$

$$(iii) 2 \tan^{-1}x = \tan^{-1} \frac{2x}{1-x^2}.$$

(i) Let  $\sin^{-1}x = \alpha$ .  $\therefore \sin \alpha = x$ ,  $\cos \alpha = \sqrt{1-x^2}$ .

$$\text{Now, } \sin 2\alpha = 2 \sin \alpha \cos \alpha = 2x\sqrt{1-x^2}.$$

$$\therefore 2\alpha = \sin^{-1}(2x\sqrt{1-x^2}).$$

Since,  $\alpha = \sin^{-1}x$ , the required result follows.

(ii) & (iii). These relations follow similarly from the corresponding values of  $\cos 2\alpha$  in terms of  $\cos \alpha$  and of  $\tan 2\alpha$  in terms of  $\tan \alpha$ . [ See Art. 43 ]

**Note.** The above three relations can also be deduced by putting  $x$  for  $y$  in the values of  $\sin^{-1}x + \sin^{-1}y$ ,  $\cos^{-1}x + \cos^{-1}y$  and  $\tan^{-1}x + \tan^{-1}y$ .

**Ex. 3.** Show that

$$(i) \quad 3 \sin^{-1}x = \sin^{-1}(3x - 4x^3).$$

$$(ii) \quad 3 \cos^{-1}x = \cos^{-1}(4x^3 - 3x).$$

$$(iii) \quad 3 \tan^{-1}x = \tan^{-1} \frac{3x - x^3}{1 - 3x^2}. \quad [C. U. 1938]$$

$$(i) \quad \text{Let } \sin^{-1}x = \theta; \text{ then } \sin \theta = x.$$

$$\text{Now, } \sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta = 3x - 4x^3.$$

$$\therefore 3\theta, \text{ i.e., } 3 \sin^{-1}x = \sin^{-1}(3x - 4x^3).$$

(ii) & (iii). These relations follow similarly from the corresponding values of  $\cos 3\theta$  in terms of  $\cos \theta$  and of  $\tan 3\theta$  in terms of  $\tan \theta$ . [See Art. 44]

**Note.** The result (iii) may also be deduced by putting  $y = z = x$  in the formula of Art. 70.

**Ex. 4.** Show that

$$2 \tan^{-1} x = \sin^{-1} \frac{2x}{1+x^2} = \cos^{-1} \frac{1-x^2}{1+x^2} = \tan^{-1} \frac{2x}{1-x^2}.$$

$$\text{Let } \tan^{-1}x = \theta, \quad \therefore \tan \theta = x.$$

$$\text{Since, } \sin 2\theta = \frac{2 \tan \theta}{1 + \tan^2 \theta} = \frac{2x}{1 + x^2}, \quad [Art. 45, Ex. 1]$$

$$\therefore 2\theta, \text{ i.e., } 2 \tan^{-1}x = \sin^{-1} \frac{2x}{1+x^2}.$$

$$\text{Since, } \cos 2\theta = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} = \frac{1 - x^2}{1 + x^2},$$

$$\text{and } \tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta} = \frac{2x}{1 - x^2},$$

the remaining relations follow similarly.

**Ex. 5.** Show that

$$\tan^{-1} \frac{a-b}{1+ab} + \tan^{-1} \frac{b-c}{1+bc} + \tan^{-1} \frac{c-a}{1+ca} = 0.$$

1st term of left side =  $\tan^{-1}a - \tan^{-1}b$  [ By Art. 69 (ii) ],

2nd ... .. =  $\tan^{-1}b - \tan^{-1}c$ ,

3rd ... .. =  $\tan^{-1}c - \tan^{-1}a$ .

Hence, adding up the three terms, the required result follows.

**Ex. 6.** Show that

$$2 \tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{4} = \tan^{-1} \frac{83}{43}.$$

Since,  $2 \tan^{-1}x = \tan^{-1} \frac{2x}{1-x^2}$ , [ See Ex. 4 ]

$$2 \tan^{-1} \frac{1}{5} = \tan^{-1} \frac{\frac{2}{5}}{1 - \frac{1}{25}} = \tan^{-1} \frac{5}{12}.$$

$$\therefore \text{left side} = \tan^{-1} \frac{5}{12} + \tan^{-1} \frac{1}{4} = \tan^{-1} \frac{\frac{5}{12} + \frac{1}{4}}{1 - \frac{5}{48}} = \tan^{-1} \frac{83}{43}.$$

**Ex. 7.** Solve

$$\sin^{-1} \frac{2a}{1+a^2} + \sin^{-1} \frac{2b}{1+b^2} = 2 \tan^{-1}x.$$

[ C. U. 1947 ]

Since,  $\sin^{-1} \frac{2x}{1+x^2} = 2 \tan^{-1}x$ , [ See Ex. 4 ]

$\therefore$  left side =  $2 \tan^{-1}a + 2 \tan^{-1}b$ .

$\therefore$  the equation reduces to

$$2 \tan^{-1}x = 2 \tan^{-1}a + 2 \tan^{-1}b.$$

$$\tan^{-1}x = \tan^{-1}a + \tan^{-1}b = \tan^{-1} \frac{a+b}{1-ab}.$$

$$x = \frac{a+b}{1-ab}.$$

Ex. 8. Solve

$$\tan^{-1} \frac{x-1}{x-2} + \tan^{-1} \frac{x+1}{x+2} = \frac{\pi}{4}.$$

$$\text{Left side} = \tan^{-1} \frac{\frac{x-1}{x-2} + \frac{x+1}{x+2}}{1 - \frac{x-1}{x-2} \cdot \frac{x+1}{x+2}} = \tan^{-1} \frac{2x^2 - 4}{x^2 - 4}.$$

$\therefore$  the equation reduces to

$$\tan^{-1} \frac{2x^2 - 4}{x^2 - 4} = \frac{\pi}{4} = \tan^{-1} 1.$$

$$\therefore \frac{2x^2 - 4}{x^2 - 4} = 1 \text{ or, } 2x^2 = 1 \text{ or, } x = \pm \frac{1}{\sqrt{2}}.$$

### Examples XII

Prove that (Ex. 1 to 17):—

1. (i)  $\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} = \frac{1}{4}\pi.$

(ii)  $\tan^{-1} x + \tan^{-1} \frac{2x}{1-x^2} = \tan^{-1} \frac{3x-x^3}{1-3x^2}.$

(iii)  $\tan^{-1} \frac{1}{7} + \tan^{-1} \frac{1}{8} + \tan^{-1} \frac{1}{15} = \cot^{-1} 3.$

2.  $\tan^{-1} \frac{3}{11} + \cot^{-1} \frac{24}{7} = \tan^{-1} \frac{1}{2}.$

3.  $\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi$   
 $= 2(\tan^{-1} 1 + \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3}).$

4. (i)  $\tan^{-1} x + \cot^{-1} (x+1) = \tan^{-1} (x^2 + x + 1).$

(ii)  $\tan^{-1} \frac{1}{p+q} + \tan^{-1} \frac{q}{p^2 + pq + 1} = \tan^{-1} \frac{1}{p}.$   
 $\tan^{-1} \frac{a-b}{1+ab} + \tan^{-1} \frac{b-c}{1+bc}.$

6.  $\tan^{-1} \frac{2}{5} + \sin^{-1} \frac{3}{5} = \tan^{-1} \frac{27}{11}.$

7.  $\tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{7} + \tan^{-1} \frac{1}{8} = \frac{1}{2}\pi.$

[ C. U. 1942 ]

8.  $2 \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{7} = \frac{1}{2}\pi.$

[ C. U. 1937 ]



9. (i)  $\sin (2 \sin^{-1} x) = 2x \sqrt{1-x^2}$ .

(ii)  $\{\cos (\sin^{-1} x)\}^2 = \{\sin (\cos^{-1} x)\}^2$ .

10.  $\cos^{-1} x = 2 \sin^{-1} \sqrt{\frac{1-x}{2}} = 2 \cos^{-1} \sqrt{\frac{1+x}{2}}$ .

11.  $\tan^{-1} \sqrt{x} = \frac{1}{2} \cos^{-1} \frac{1-x}{1+x}$ . [ C. U. 1943 ]

12.  $\sin^{-1} \sqrt{\frac{x-b}{a-b}} = \cos^{-1} \sqrt{\frac{a-x}{a-b}} = \tan^{-1} \sqrt{\frac{x-b}{a-x}}$ .

13.  $\tan^{-1} \frac{a-b}{1+ab} + \tan^{-1} \frac{b-c}{1+bc} + \tan^{-1} \frac{c-a}{1+ca}$   
 $= \tan^{-1} \frac{a^2-b^2}{1+a^2b^2} + \tan^{-1} \frac{b^2-c^2}{1+b^2c^2} + \tan^{-1} \frac{c^2-a^2}{1+c^2a^2}$

14.  $\sec^2 (\tan^{-1} 2) + \operatorname{cosec}^2 (\cot^{-1} 3) = 15$ .

15.  $\cot^{-1} (\tan 2x) + \cot^{-1} (-\tan 3x) = x$ .

16.  $\sin^{-1} \frac{4}{5} + \sin^{-1} \frac{5}{13} + \sin^{-1} \frac{16}{65} = \frac{1}{2}\pi$ . [ C. U. 1941 ]

17.  $4 (\cot^{-1} 3 + \operatorname{cosec}^{-1} \sqrt{5}) = \pi$ . [ C. U. 1939 ]

18. If  $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \pi$ , show that  
 $x + y + z = xyz$ .

19. If  $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \frac{1}{2}\pi$ , show that  
 $yz + zx + xy = 1$ .

20. If  $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$ , show that  
 $x^2 + y^2 + z^2 + 2xyz = 1$ .

21. If  $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \pi$ , show that  
 $x \sqrt{1-x^2} + y \sqrt{1-y^2} + z \sqrt{1-z^2} = 2xyz$ .

22. Find the values of

(i)  $\sin (\sin^{-1} \frac{1}{2} + \cos^{-1} \frac{1}{2})$ . [ C. U. 1935 ]

(ii)  $\cot (\tan^{-1} a + \cot^{-1} a)$ .

(iii)  $\tan \left( \frac{1}{2} \sin^{-1} \frac{2x}{1+x^2} + \frac{1}{2} \cos^{-1} \frac{1-y^2}{1+y^2} \right)$ .

23. If  $\tan^{-1}y = 4 \tan^{-1}x$ , find  $y$  as an algebraic function of  $x$ .

24. If  $\tan^{-1}x, \tan^{-1}y, \tan^{-1}z$  are in A.P., find out the algebraic relation between  $x, y, z$ . If in addition,  $x, y, z$  are also in A.P., prove that  $x = y = z$ . [ $y \neq 0, 1$  or  $-1$ ]

25. Solve the following equations :

$$(i) \tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1} \frac{a}{y^2}.$$

$$(ii) \tan^{-1} \frac{2x}{1-x^2} = \sin^{-1} \frac{2a}{1+a^2} - \cos^{-1} \frac{1-b^2}{1+b^2}.$$

$$(iii) \tan(\cos^{-1}x) = \sin(\tan^{-1}2).$$

$$(iv) \tan^{-1} \frac{1-x}{1+x} = \frac{1}{2} \tan^{-1} x.$$

$$(v) \tan^{-1} \frac{x-1}{x+1} + \tan^{-1} \frac{2x-1}{2x+1} = \tan^{-1} \frac{23}{36}.$$

$$(vi) \sin^{-1}x + \sin^{-1}2x = \frac{\pi}{3}.$$

$$(vii) \sin^{-1}x + \sin^{-1}(1-x) = \cos^{-1}x.$$

$$(viii) \tan^{-1}(x-1) + \tan^{-1}x + \tan^{-1}(x+1) = \tan^{-1}3x.$$

$$(ix) \tan^{-1} \frac{2x}{1-x^2} + \cot^{-1} \frac{1-x^2}{2x} = \frac{\pi}{3}.$$

$$(x) \cot^{-1}(x-1) + \cot^{-1}(x-2) + \cot^{-1}(x-3) = 0.$$

26. Show that

$$(i) \cot^{-1} \frac{xy+1}{x-y} + \cot^{-1} \frac{yz+1}{y-z} + \cot^{-1} \frac{zx+1}{z-x} = 0.$$

$$(ii) \tan(\tan^{-1}x + \tan^{-1}y + \tan^{-1}z) \\ = \cot(\cot^{-1}x + \cot^{-1}y + \cot^{-1}z).$$

$$(iii) \tan^{-1}(\cot x) + \cot^{-1}(\tan x) = \pi - 2x.$$

### Miscellaneous Examples I

1. If  $3 \sin \theta + 4 \cos \theta = 5$ , show that  $\tan \theta = \frac{3}{4}$ .
2. If  $a^2 \sec^2 x - b^2 \tan^2 x = c^2$ , find  $\operatorname{cosec} x$ .
3. If  $x = r \cos \theta \cos \phi$ ,  $y = r \cos \theta \sin \phi$ ,  $z = r \sin \theta$ , show that  $x^2 + y^2 + z^2 = r^2$ .

4. If  $\sin \theta = \frac{x-y}{x+y}$ , show that  $\tan \left( \frac{\pi}{4} - \frac{\theta}{2} \right) = \pm \sqrt{\frac{y}{x}}$ .

5. If  $x = r \sin (\theta + 45^\circ)$  and  $y = r \sin (\theta - 45^\circ)$ , then  $x^2 + y^2 = r^2$ .

6. If  $\cos (a + \beta) \sin (\gamma + \theta) = \cos (a - \beta) \sin (\gamma - \theta)$ , then  $\tan \theta = \tan a \tan \beta \tan \gamma$ .

Show that (Ex. 7 to 9) :—

7.  $(\cos x - \cos y)^2 + (\sin x - \sin y)^2 = 4 \sin^2 \frac{x-y}{2}$ .

8.  $\sin A + \sin B + \sin C - \sin (A + B + C)$   
 $= 4 \sin \frac{A+B}{2} \sin \frac{B+C}{2} \sin \frac{C+A}{2}$ .

9.  $4 \sin \frac{A+B+C}{2} \sin \frac{B+C-A}{2} \sin \frac{C+A-B}{2} \sin \frac{A+B-C}{2}$   
 $= 1 - \cos^2 A - \cos^2 B - \cos^2 C + 2 \cos A \cos B \cos C$ .

10. If  $\tan \beta = \frac{2 \sin \alpha \sin \gamma}{\sin (\alpha + \gamma)}$ , then  $\tan \alpha$ ,  $\tan \beta$ ,  $\tan \gamma$  are in harmonical progression.

11. If  $\alpha + \beta + \gamma = (2n+1) \frac{\pi}{2}$ , then

- (i)  $\tan \beta \tan \gamma + \tan \gamma \tan \alpha + \tan \alpha \tan \beta = 1$ .

- (ii)  $\sin 2\alpha + \sin 2\beta + \sin 2\gamma = \pm 4 \cos \alpha \cos \beta \cos \gamma$ .

12. If the angles  $A, B, C$  be in A.P., then

$$\frac{\sin A - \sin C}{\cos C - \cos A} = \frac{\cos B}{\sin B}$$

13. If  $\operatorname{cosec} 2A + \operatorname{cosec} 2B + \operatorname{cosec} 2C = 0$ , show that  
 $\tan A + \tan B + \tan C + \cot A + \cot B + \cot C = 0$ .
14. If  $\tan \alpha = \frac{a \sin \beta}{1 - a \cos \beta}$  and  $\tan \beta = \frac{b \sin \alpha}{1 - b \cos \alpha}$ ,  
 then  $\frac{\sin \alpha}{\sin \beta} = \frac{a}{b}$ .
15. Show that  
 $\tan \theta + 2 \tan 2\theta + 4 \tan 4\theta + 8 \cot 8\theta = \cot \theta$ .
16. If  $\cos(\theta - \psi) \cos \phi = \cos(\theta - \phi + \psi)$ , then  $\tan \theta, \tan \phi$ ,  
 $\tan \psi$  are in harmonical progression.
17. If  $1 + \cos(y - z) + \cos(z - x) + \cos(x - y) = 0$ , show  
 that either  $(y - z)$ , or  $(z - x)$ , or  $(x - y)$  is an odd multiple of  $\pi$ .
18. If  $\sin \theta + \sin \phi = \sqrt{3}(\cos \phi - \cos \theta)$ , show that  
 $\sin 3\theta + \sin 3\phi = 0$ .
19. Eliminate  $\alpha$  and  $\beta$  from  
 $\sin \alpha + \sin \beta = a, \cos \alpha + \cos \beta = b, \cos(\alpha - \beta) = c$ .
20. If  $A + B + C = \pi$ , prove that  
 (i)  $\tan B \tan C + \tan C \tan A + \tan A \tan B = 1 + \sec A \sec B \sec C$ .  
 (ii)  $\cot A + \cot B + \cot C = \cot A \cot B \cot C$   
 $+ \operatorname{cosec} A \operatorname{cosec} B \operatorname{cosec} C$ .
21. If  $A + B + C = \pi$ , and if  
 $\sin^2 A + \sin^2 B + \sin^2 C = \sin B \sin C + \sin C \sin A$   
 $+ \sin A \sin B$ , then  $A = B = C$ .
22. If  $A, B, C$  be the angles of a triangle, and if  
 $\cot A + \cot B + \cot C = \sqrt{3}$ , show that the triangle  
 is equilateral.
23. If  $\sec ax + \sec bx = 0$ , show that the values of  $x$  form  
 two series in A.P.

## CHAPTER XIII

### LOGARITHMS

#### 72. Definition of Logarithm.

Logarithm of a number with respect to a given base is the index of the power to which the base is to be raised in order to give the number.

Mathematically, if  $a^x = N$ , then ' $x$ ' is the index of the power to which ' $a$ ' (which is called the base) is raised to give ' $N$ '. Hence, by definition, ' $x$ ' is the logarithm of ' $N$ ' with respect to the base ' $a$ ' and it is usually written as  $x = \log_a N$ .

As a numerical example,  $\log_2 8 = 3$ , for  $2^3 = 8$  i.e., 3 is the power to which 2 is to be raised to give 8. Again, since  $3^4 = 81$ ,  $4 = \log_3 81$ .

Any result involving indices can be expressed as a result in logarithm, and *vice versa*.

For example,

$$\begin{aligned} \text{if } p^q &= r, \quad \text{then, } q = \log_p r, \\ \text{if } m^n &= z^k, \text{ then } n = \log_m (z^k), \\ \text{or, } k &= \log_z (m^n). \end{aligned}$$

Similarly, if  $\log_y x = z$ ,  
then  $y^z = x$ .

It should be noted that the logarithms of the same number with respect to different bases will be different ; for example, to get the same number 64, we must raise 2 to the power 6, whereas we are to raise 4 to the power 3 and 8 to the power 2 only ; hence  $\log_2 64 = 6$ ,  $\log_4 64 = 3$ ,  $\log_8 64 = 2$ .

Thus, so long as the base is not stated, logarithm of a number has no meaning.

### 73. Special Cases.

We know from Algebra that if  $a$  be any real finite quantity, other than zero, then  $a^0 = 1$ .

Hence,  $\log_a 1 = 0$  ; in other words.

(i) *logarithm of 1 with respect to any finite quantity (other than zero) as base, is zero.*

Again,  $a$  being any quantity,  $a^1 = a$ .

Hence,  $1 = \log_a a$  ; in other words,

(ii) *logarithm of any number with respect to itself as base is unity.*

**Note 1.** If  $a^x = 0$ , then  $x = -\infty$  if  $a > 1$ , and  $x = +\infty$  if  $a < 1$ .

Thus, we have  $\log_a 0 = \mp \infty$  according as  $a >$  or  $< 1$ . Hence, *logarithm of zero to a base greater than unity is minus infinity, and to a base less than unity is plus infinity.*

**Note 2** Since the equation  $a^x = -n$  ( $a$  and  $n$  being real positive quantities), cannot be satisfied by any real value of  $x$ , whether positive or negative, provided we consider the principal value\* only of  $a^x$ , therefore, *logarithm of a negative quantity (in a system of logarithms whose base is a real positive quantity) must be imaginary.*

### 74. Fundamental formulæ in logarithms.

From the definition it is clear that logarithms are but indices in another form. Hence, corresponding to the three fundamental results in the theory of indices in Algebra, namely that if  $a, x, y$  be any real quantities,

$$(i) \quad a^x \times a^y = a^{x+y},$$

$$(ii) \quad a^x \div a^y = a^{x-y}, \text{ and}$$

$$(iii) \quad (a^x)^y = a^{xy},$$

we get three fundamental laws of logarithms which are given below.

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\* See a treatise on Higher Trigonometry.

$$(i) \log_a (m \times n) = \log_a m + \log_a n.$$

In other words, *logarithm of the product of two quantities is equal to the sum of their logarithms taken separately, base remaining the same always.*

**Proof.** Put  $\log_a m = x$ ,  $\log_a n = y$   
and  $\log_a (m \times n) = z$ .

Then from definition,

$$a^x = m, a^y = n \text{ and } a^z = m \times n = a^x \times a^y = a^{x+y},$$

so that,  $z = x + y$ .

Replacing the values,

$$\log_a (mn) = \log_a m + \log_a n.$$

**Cor.**  $\log_a (m.n.p.\dots) = \log_a m + \log_a n + \log_a p + \dots$

$$(ii) \log_a \left( \frac{m}{n} \right) = \log_a m - \log_a n.$$

In other words, *logarithm of the quotient of two numbers is equal to the difference of their logarithms (logarithm of the numerator minus logarithm of the denominator).*

**Proof.** Put  $\log_a m = x$ ,  $\log_a n = y$

$$\text{and } \log_a \left( \frac{m}{n} \right) = z.$$

Then, from definition,

$$a^x = m, a^y = n$$

$$\text{and } a^z = \frac{m}{n} = \frac{a^x}{a^y} = a^{x-y},$$

so that

$$z = x - y,$$

or replacing the values,

$$\log_a \left( \frac{m}{n} \right) = \log_a m - \log_a n.$$

(iii)  $\log_a (m)^n = n \log_a m$ .

Or, *logarithm of a power of a number is the product of the power and the logarithm of the number.*

**Proof.** Put  $\log_a m = x$ , and  $\log_a (m)^n = z$ .

Then, by definition,

$$a^x = m \quad \text{and}$$

$$a^z = (m)^n = a^{nx}.$$

$$\therefore z = nx,$$

or replacing the values,

$$\log_a (m)^n = n \log_a m.$$

**Ex. 1.** Reduce to a simple form  $\log_a \frac{x^p y^q}{z^s}$ .

$$\begin{aligned} \log_a \frac{x^p y^q}{z^s} &= \log_a (x^p y^q) - \log_a (z^s) \\ &= \log_a x^p + \log_a y^q - \log_a z^s \\ &= p \log_a x + q \log_a y - s \log_a z. \end{aligned}$$

**Ex. 2.** Simplify  $\log_{10} \sqrt[3]{\frac{25}{88}}$ .

$$\begin{aligned} \log_{10} \sqrt[3]{\frac{25}{88}} &= \log_{10} \left( \frac{5^2}{8 \cdot 11} \right)^{\frac{1}{3}} = \frac{1}{3} \log_{10} \frac{5^2}{2^3 \cdot 11} \\ &= \frac{1}{3} \log_{10} \frac{10^2}{2^5 \cdot 11} \\ &= \frac{1}{3} [ \log_{10} 10^2 - \log_{10} (2^5 \cdot 11) ] \\ &= \frac{1}{3} [ 2 \log_{10} 10 - (\log_{10} 2^5 + \log_{10} 11) ] \\ &= \frac{1}{3} [ 2 - 5 \log_{10} 2 - \log_{10} 11 ]. \end{aligned}$$

## 75. Change of base.

There is a fourth standard formula whereby logarithms of numbers with respect to one base being given, those with respect to a different base may be obtained. The formula is

$$\log_a m = \log_b m \times \log_a b.$$



**Proof.** Put  $\log_a m = x$ ,  $\log_b m = y$  and  $\log_a b = z$ .

Then, from definition,

$$a^x = m, b^y = m, a^z = b.$$

Hence,  $x^x = m = b^y = (a^z)^y = a^{yz}$ ,

$$\text{or, } x = yz.$$

Replacing the values,

$$\log_a m = \log_b m \times \log_a b.$$

**Cor. 1.** In the above result, put  $m = a$ . Then remembering that  $\log_a a = 1$ , we get

$$\log_b a \times \log_a b = 1.$$

Since, the above relation is very important, we add here an *independent proof* of it.

Let  $\log_b a = x$ , and  $\log_a b = y$ .

Then,  $b^x = a$  and  $a^y = b$ .

$$\therefore a = b^x = (a^y)^x = a^{xy}. \quad \therefore xy = 1,$$

$$\text{i.e., } \log_b a \times \log_a b = 1,$$

$$\text{or, } \log_b a = \frac{1}{\log_a b}.$$

**Cor. 2.** The result of the above article may be written with the help of Cor. 1, in the form

$$\log_a m = \log_b m / \log_b a.$$

Thus, if logarithms of both  $m$  and  $a$  with respect to  $b$  be known, logarithm of  $m$  with respect to  $a$  is obtained.

## 76. Common system of logarithms.

For all practical purposes wherever logarithms are used for numerical calculations, the base is usually taken as 10. Logarithms of numbers with respect to the base 10 are

referred to as the *Common system* of logarithms. The advantage of the common system of logarithms for practical applications will be clear presently, from the Article 77, Theorems I & II.

**Note.** In higher mathematics, for *theoretical* investigations, another quantity 'e' (defined in books of Algebra), whose value is nearly 2.718... is used as the base of logarithms, and logarithms to this base e are called *Napierian logarithms*.

With the help of the logarithmic series established in books on Algebra, Napierian logarithms of numbers are tabulated. The factor  $\log_{10} e$  which is known as the *modulus of the common system*, applied to the Napierian logarithms will convert them to common logarithms (See Art. 75). Thus, a table of common logarithms is prepared.

Henceforth, we shall proceed with the consideration of the common system of logarithms, and the base being understood to be 10, will not be written.

### 77. Characteristic and Mantissa of common logarithms.

It is only in very few cases that the logarithm of a number is integral. In most cases, however, the logarithm of a number is partly integral and partly fractional (or decimal).

**Def.** The integral portion of the logarithm of a number is called the *characteristic*, and the decimal portion is called the *mantissa*.

In case the logarithm of a number is negative, and partly integral and partly decimal, the decimal portion, *i.e.*, the mantissa is always kept positive by altering the integral part, *i.e.*, the characteristic suitably. Thus, the *mantissa part of the logarithm of a number is always positive*. For instance, if the logarithm of a number is  $-2.3$ , we write it as  $-3 + .7$  and call  $-3$  as the characteristic and  $.7$  (and not  $-.3$ ) as the mantissa.  $-3 + .7$  is often abbreviated in the form  $\bar{3}.7$ .

**Theorem I.** *The characteristic of the common logarithm of (i) any number greater than 1 is positive, and numerically one less than the number of digits in the integral part of the quantity whose logarithm is sought; and (ii) of any positive\* number less than 1, is negative, and numerically one greater than the number of zeros immediately after the decimal point in the quantity whose logarithm is wanted.*

(i) Let the number be greater than unity.

Any number, say 7'209, which consists of 1 digit only in its integral part, lies between 1 and 10.

Now,  $10^0 = 1$  and  $10^1 = 10$ .

Hence, if  $10^x = 7'209$ , clearly  $x$  must be greater than 0 and less than 1.

Thus,  $\log 7'209$  must be between 0 and 1, i.e., of the form 0'..., having its characteristic 0.

Similarly, numbers of the type 53'0528, which consists of 2 digits in their integral parts, must lie between 10 and 100 i.e., between  $10^1$  and  $10^2$ .

Hence, the index to which 10 should be raised to give 53'0528 must be greater than 1 and less than 2, i.e.,  $\log 53'0528$  must be of the form 1'... having the characteristic 1.

$\log 10$  is 1, and 10 also falls in this category of two digits.

In the same way, a number which has  $n$  digits in its integral part lies between  $10^{n-1}$  (which also has  $n$  digits) and  $10^n$  (which has  $n+1$  digits). Thus, the logarithms of such numbers must lie between  $n-1$  and  $n$ , i.e.,  $(n-1) + \text{some positive proper fraction}$ . Hence, the characteristic in such cases is  $n-1$ .

Hence, the result.

\*Logarithms of negative numbers are easily seen to be imaginary, for there is no real power, positive or negative, to which 10 may be raised to give a negative result. [ See Note 2, Art. 73 ]

(ii) Let the number be positive, and less than 1 (*i.e.*, between 0 and 1).

We notice that

$$\begin{aligned} 10^0 &= 1 \\ 10^{-1} &= \frac{1}{10} = .1 \\ 10^{-2} &= \frac{1}{100} = .01 \\ 10^{-3} &= \frac{1}{1000} = .001 \\ 10^{-4} &= \frac{1}{10000} = .0001 \\ \text{etc.} \quad \text{etc.} \quad \text{etc.} \end{aligned}$$

Now, a number less than 1, with no zero immediately after the decimal point, like .3015, must be greater than .1 and less than 1; hence, the power to which 10 must be raised to give such a number must lie between -1 and 0, *i.e.*, = -1 + a positive proper fraction. Hence, such numbers have the characteristic of their logarithms = -1.

A decimal number with one zero immediately after the decimal point, like .078005, lies between .01 and .1 which are respectively equal to  $10^{-2}$  and  $10^{-1}$ .

Hence, if  $10^x = .078005$ ,  $x$  must lie between -1 and -2 *i.e.*,  $x$  is of the form  $-1 \cdots \cdots$ . Writing the decimal part of  $x$  positively, in the form  $-2 + \cdots \cdots$ , we notice that the integral part of  $x$ , *i.e.*, the characteristic of the logarithm of .078005 is -2.

Similarly, the logarithms of numbers between .01 and .001 (*i.e.*,  $10^{-2}$  and  $10^{-3}$ ) which must have two zeros after the decimal point, lie between -2 and -3, *i.e.*, are of the form  $-2 \cdots \cdots = -3 + \cdots \cdots$ , and so the characteristic in such cases is -3,

and so on.

Hence the result.

**Theorem II.** *All numbers, formed of the same digits in the same order, differing only in the positions of their decimal points, have the mantissæ of their logarithms same.*

This will be clear from an example. Let us take the numbers 835107, 835107000, 83'5107, '835107, '000835107 and 8351'07.

$$\begin{aligned}\text{Now, } \log 835107000 &= \log (835107 \times 1000) \\ &= \log 835107 + \log 1000 \\ &= \log 835107 + '3.\end{aligned}$$

$$\begin{aligned}\text{Again, } \log 83'5107 &= \log \frac{835107}{10000} \\ &= \log 835107 - \log 10000 \\ &= \log 835107 - 4.\end{aligned}$$

$$\log '835107 = \log \frac{835107}{1000000} = \log 835107 - 6.$$

$$\log '000835107 = \log \frac{835107}{10^9} = \log 835107 - 9.$$

$$\log 8351'07 = \log \frac{835107}{100} = \log 835107 - 2.$$

Thus, the logarithms of all the numbers here differ from the logarithm of 835107 by a whole number in each case and so must have their decimal parts, *i.e.*, their mantissæ the same as that of  $\log 835107$ .

In fact, numbers formed of the same digits in the same order differing only in the position of their decimal points, must have their ratios equal to an integral power of 10 and so must have their logarithms differing only by a whole number.

Hence the result.

The two theorems above given show that (i) the characteristic of the logarithm of a number can be found by a simple glance at the number and (ii) that for the mantissa part of the logarithm of a number, we need only take into

account the digits of which the number is formed, without taking any notice of the position of the decimal point in it.

In logarithmic tables, only the mantissæ of the logarithms of numbers are therefore given.

These constitute the special advantages of the common system of logarithms.

### 78. Examples worked out.

Ex. 1. *Simplify*

$$\log \frac{\sqrt[4]{5} \cdot \sqrt[10]{2}}{\sqrt[3]{(18 \cdot \sqrt{2})}}, \text{ and find its value, given}$$

$$\log 2 = \cdot 30103 \text{ and } \log 3 = \cdot 4771213.$$

$$\text{The given exp.} = \log \frac{5^{\frac{1}{4}} \cdot 2^{\frac{1}{10}}}{(18 \cdot 2^{\frac{1}{2}})^{\frac{1}{3}}}$$

$$= \log \frac{10^{\frac{1}{4}} \cdot 2^{\frac{1}{10}}}{2^{\frac{1}{3}} (2 \cdot 3^2 \cdot 2^{\frac{1}{2}})^{\frac{1}{3}}} = \log \frac{10^{\frac{1}{4}} \cdot 2^{\frac{1}{10}}}{2^{\frac{1}{3}} \cdot 2^{\frac{1}{3}} \cdot 3^{\frac{2}{3}} \cdot 2^{\frac{1}{6}}}$$

$$= \log \frac{10^{\frac{1}{4}}}{2^{\frac{1}{3} + \frac{1}{6}} \cdot 3^{\frac{2}{3}}} = \log 10^{\frac{1}{4}} - \log (2^{\frac{1}{2}} \times 3^{\frac{2}{3}})$$

$$= \frac{1}{4} \log 10 - (\log 2^{\frac{1}{2}} + \log 3^{\frac{2}{3}})$$

$$= \frac{1}{4} \log 10 - \frac{1}{2} \log 2 - \frac{2}{3} \log 3$$

and its value is

$$\begin{aligned} & \frac{1}{4} \cdot 1 - \frac{1}{2} \cdot 0 (\cdot 30103) - \frac{2}{3} (\cdot 4771213) \\ &= \cdot 25 - \cdot 1956695 - \cdot 3180809 \\ &= -1 + \cdot 7362496 \\ &= \bar{1} \cdot 7362496. \end{aligned}$$

**Note.**  $\log 5 = \log \frac{10}{2} = \log 10 - \log 2 = 1 - \log 2$  and hence  $\log 5$  is deducible from  $\log 2$ .

Ex. 2. *Prove that*

$$7 \log \frac{10}{3} - 2 \log \frac{5}{2} + 3 \log \frac{81}{10} = \log 2.$$

The left-hand expression

$$\begin{aligned}
 &= \log \left(\frac{10}{9}\right)^7 - \log \left(\frac{25}{24}\right)^2 + \log \left(\frac{81}{80}\right)^3 \\
 &= \log \frac{\left(\frac{10}{9}\right)^7 \times \left(\frac{81}{80}\right)^3}{\left(\frac{25}{24}\right)^2} \\
 &= \log \left\{ \left(\frac{10}{3^2}\right)^7 \times \left(\frac{3^4}{10 \times 2^3}\right)^2 \times \left(\frac{3 \times 2^3 \times 2^2}{10^2}\right)^3 \right\} \\
 &= \log \left( \frac{10^7}{3^{14}} \times \frac{3^{12}}{10^8 \times 2^6} \times \frac{3^9 \times 2^{10}}{10^6} \right) \\
 &= \log 2.
 \end{aligned}$$

*Alternative method :*

Left side

$$\begin{aligned}
 &= 7(\log 10 - \log 9) - 2(\log 25 - \log 24) + 3(\log 81 - \log 80) \\
 &= \{\log (5 \times 2) - \log 3^2\} - 2\{\log 5^2 - \log (3 \times 2^3)\} \\
 &\quad + 3\{\log 3^4 - \log (5 \times 2^4)\} \\
 &= 7\{\log 5 + \log 2 - 2 \log 3\} - 2\{2 \log 5 - \log 3 - 3 \log 2\} \\
 &\quad + 3\{4 \log 3 - \log 5 - 4 \log 2\} \\
 &= \log 2.
 \end{aligned}$$

**Ex. 3.** Find the number of digits in  $4^{15}$ , having given  $\log 2 = .30103$ .

We have

$$\begin{aligned}
 \log 4^{15} &= \log 2^{30} = 30 \log 2 \\
 &= 30 \times .30103 = 9.0309.
 \end{aligned}$$

Hence, since the characteristic of  $\log 4^{15}$  is 9,  $4^{15}$  must consist of 10 digits.

**Ex. 4.** Find approximately the 7<sup>th</sup> root of 3.528, having given  $\log 2 = .30103$ ,  $\log 3 = .4771213$ ,  $\log 7 = .8450980$  and  $\log 1197.342 = 3.0782184$ .

$$\text{Let } x = (3.528)^{\frac{1}{7}} = \left( \frac{7^2 \times 3^2 \times 2^2}{10^3} \right)^{\frac{1}{7}}$$

$$\begin{aligned}
 \text{then } \log x &= \frac{1}{7} [2 \log 7 + 2 \log 3 + 3 \log 2 - 3 \log 10] \\
 &= \frac{1}{7} [2 \times .8450980 + 2 \times .4771213 + 3 \times .30103 - 3] \\
 &= .0782184 \text{ nearly.}
 \end{aligned}$$

Now,  $\log 1197'342 = 3'0782184$ .

$\therefore \log 1'197342 = '0782184$ , having characteristic 0, but mantissa same as that of  $\log 1197'342$ .

Hence,  $x = 1'197342$  approximately.

**Ex. 5.** Obtain an approximate numerical solution of  $2^x \cdot 3^{2x} = 100$ , having given  $\log 2 = '30103$ ,  $\log 3 = '47712$ .

We have

$$2^x \cdot 3^{2x} = 10^2.$$

$$\therefore \log (2^x \cdot 3^{2x}) = \log 10^2,$$

$$\text{i.e., } x \log 2 + 2x \log 3 = 2 \log 10 = 2.$$

$$\therefore x = \frac{2}{\log 2 + 2 \log 3} = \frac{2}{'30103 + 2 \times '47712} \\ = 1'5933 \text{ nearly.}$$

**Note.** Equations of this type are called **Exponential Equations**.

### Examples XIII(a)

[ Use the values :  $\log 2 = '30103$ ,  $\log 3 = '4771213$ ,  
 $\log 7 = '8450980$  when required ]

1. Find the logarithm of (i) 1728 to the base  $2\sqrt{3}$ ,  
(ii)  $\cos^3 \alpha$  to the base  $\sec \alpha$ .

2. Find  $\log_{.01} 10000$ .

3. Show that  $\log_{10} 2$  lies between  $\frac{1}{3}$  and  $\frac{1}{2}$ .

[ C. U. 1926 ]

4. Prove that

$$(i) \log_a m \times \log_b n = \log_b m \times \log_a n.$$

$$(ii) \log_2 \log_2 \log_2 16 = 1.$$

5. If  $\log_e m + \log_e n = \log_e (m + n)$ , find  $m$  as a simple function of  $n$ .

6. Prove that if a series of numbers be in G.P., their logarithms are in A.P.



7. Prove that

$$2 \log a + 2 \log a^2 + 2 \log a^3 + \dots + 2 \log a^n \\ = n(n+1) \log a.$$

8. If  $x$  is positive and less than unity, show that  
 $\log(1+x) + \log(1+x^2) + \log(1+x^4) + \log(1+x^8) + \dots$  to  $\infty$   
 $= -\log(1-x).$

9. Simplify

(i)  $\log_2 \sqrt{6} + \log_2 \sqrt{\frac{2}{3}}.$

(ii)  $\frac{\log \sqrt{27} + \log 8 - \log \sqrt{1000}}{\log 1.2}.$

10. Find  $\log(00225)^{\frac{1}{3}}$  and  $\log(\sqrt[5]{\frac{1}{2}})^{-\frac{1}{3}}.$

11. Prove that

(i)  $\log_a b \times \log_b c \times \log_c a = 1.$

(ii)  $\log_a x = \log_b x \times \log_c b \times \log_d c \dots \times \log_n m \times \log_a n.$

12. Show that

(i)  $7 \log \frac{1}{12} + 5 \log \frac{25}{4} + 3 \log \frac{81}{80} = \log 2.$

(ii)  $7 \log \frac{1}{6} + 6 \log \frac{8}{9} + 5 \log \frac{2}{3} + \log \frac{32}{27} = \log 3.$

13. Extract the fifth root of 84, having given

$$\log 2425805 = 6.3848559.$$

14. Calculate  $(0020736)^{\frac{1}{7}}$ , having given

$$\log 41369 = 4.6166750.$$

15. Simplify

(i)  $\log \sqrt[7]{\frac{8^{\frac{1}{2}} \times 14^{\frac{1}{3}}}{\sqrt{72} \times \sqrt[5]{60}}}.$

(ii)  $\sqrt[3]{\frac{7.2 \times 6.3}{62.5}},$  having given

$$\log 898665 = 5.9535977.$$

16. Find the value of  $64 \{1 - (1.05)^{-20}\}$ , having given  
 $\log 24121 = 4.382394$ .

17. Find the number of digits in (i)  $2^{40}$ , (ii)  $3^{11}$ ,  
 (iii)  $(540)^9$ .

18. Find the number of zeros after the decimal point  
 before the first significant digit in the expressions

$$(i) (.024)^{15}, \quad (ii) \left(\frac{1}{4.05}\right)^8, \quad (iii) (.0259)^{50}.$$

19. Solve the equations:—

$$(i) 3^x = 2, \quad (ii) 3^{x-4} = 7.$$

$$(iii) 5^{6x} + 7^{x+2} = 3^{2x-3}.$$

$$(iv) \left. \begin{aligned} 2^x &= 3^y \\ 2^{y+1} &= 3^{x-1} \end{aligned} \right\} \quad (v) \left. \begin{aligned} 7^{x+y} \times 3^{2x+y} &= 9 \\ 3^{x-y} + 2^{x-2y} &= 3^x \end{aligned} \right\}$$

20. (i) If  $\log(x^2 y^3) = a$ ,  $\log\left(\frac{x}{y}\right) = b$ , find  $\log x$  and  $\log y$ .

(ii) If  $a^2 + b^2 = 7ab$ , show that

$$\log\left\{\frac{1}{3}(a+b)\right\} = \frac{1}{3}(\log a + \log b).$$

21. If  $\frac{\log x}{y-z} = \frac{\log y}{z-x} = \frac{\log z}{x-y}$ , show that  $x^x y^y z^z = 1$ .

22. Why is  $\log(1+2+3) \neq \log 1 + \log 2 + \log 3$ ?

23. If  $a, b, c, \dots$  be in G.P., show that

$$\log_a x, \log_b x, \log_c x, \dots \text{ are in H.P.}$$

24. If  $xy^{l-1} = a$ ,  $xy^{m-1} = b$ ,  $xy^{n-1} = c$ , prove that

$$(m-n) \log a + (n-l) \log b + (l-m) \log c = 0.$$

25. If  $\frac{x(y+z-x)}{\log x} = \frac{y(z+x-y)}{\log y} = \frac{z(x+y-z)}{\log z}$ , show that

$$y^x z^y = z^x x^z = x^y y^x.$$

## 79. Tables of Logarithms and Trigonometrical ratios.

Several mathematical tables correct up to five places of decimals are given at the end of the book. An explanation of the tables is given below.

Table I gives the common logarithm of all numbers from 1 to 10000, *i.e.*, those which consist of 4 digits or less. The tabulated quantities are the mantissæ only, correct to five places, with the decimal point dropped. The characteristic is to be supplied according to the rule given in Art. 77. The main body of the table gives logarithms (mantissa part) of numbers of 3 digits, and the mean difference table at the side supplies the increment in the mantissa due to the fourth digit. This increment is written, in order to save space, giving the significant digits only, which are to be supplied with the necessary number of zeros to make up 5 places (here the table being a five-figure table). Thus, '00024 will be written as 24 only in the difference table. As an example, to find  $\log 2'697$ , we notice from the table that the mantissa for  $\log 269$  is '42975, and along the same row, the difference table gives 115 under the heading 7. This means that for 7 in the fourth place of the number (*i.e.*, for the number 2697) the increment in the mantissa will be '00115. Hence,  $\log 2697$  will have its mantissa '42975 + '00115 = '43090. Again,  $\log 2'697$  has the same mantissa but its characteristic is 0. Thus,  $\log 2'697 = 0'43090$ .

Table II gives ordinary sines and cosines (usually referred to as *natural sines and cosines*) of all angles from  $0^\circ$  to  $90^\circ$  at intervals of  $1'$ , sines being given from the left side of the top towards the right and downwards, and cosines being given from the right side of the bottom towards the left and upwards. The table is arranged in such a way that the sine of any angle given is the same as the cosine of exactly the complementary angle, and it is on this arrangement that a single table serves as a sine as well as a cosine table. The main portion of the table gives sines or cosines of angles at intervals of  $10'$ , and the difference

table at the side gives changes in the value of the sine or cosine for changes in minutes in the angles. It should be remembered that as an angle increases from  $0^\circ$  to  $90^\circ$ , its sine increases from 0 to 1 whereas its cosine decreases from 1 to 0. Hence, *the changes given in the difference table are to be added in case of sines and subtracted in case of cosines* for the increased number of minutes in the angles. Moreover, as in Table I, the numbers in the difference table are to be made up to five places of decimals by supplying the requisite number of zeros before it. For example, using the table,  $\sin 53^\circ 23' = .80212 + .00052 = .80264$  and  $\cos 20^\circ 42' = .86892 - .00029 = .86863$ .

Table III similarly gives natural tangents and cotangents of angles from  $0^\circ$  to  $90^\circ$ , obtained at intervals of  $1'$  with the help of the difference table. The quantities in the difference table, being made up into five figures, are to be *added in case of tangents and subtracted in case of cotangents* for increased number of minutes in the angle.

Table IV gives logarithmic sines and logarithmic cosines of all angles from  $0^\circ$  to  $90^\circ$  at intervals of  $1'$  (with the aid of the difference table). Logarithmic sine of angle  $\theta$ , written as  $L \sin \theta$  means  $10 + \log \sin \theta$ , and similarly, logarithmic cosine of  $\theta$ , written as  $L \cos \theta$  means  $10 + \log \cos \theta$ . In taking logarithms of trigonometrical ratios of angles, it may be noted that sines and cosines of angles are numerically less than unity, and tangents of angles between  $0^\circ$  and  $45^\circ$  as also cotangents of angles between  $45^\circ$  and  $90^\circ$  are less than unity. Hence, logarithms of these quantities are negative. To avoid using negative values in the tables, *logarithms of trigonometrical ratios are always tabulated after adding 10 to them*. Thus, the table gives  $L \sin \theta$  and  $L \cos \theta$  (and not  $\log \sin \theta$  and  $\log \cos \theta$ ).

Table V gives logarithmic tangents (*i.e.*,  $L \tan \theta = 10 + \log \tan \theta$ ) and logarithmic cotangents (*i.e.*,  $L \cot \theta = 10 + \log \cot \theta$ ) of all angles from  $0^\circ$  to  $90^\circ$ , obtained at intervals of  $1'$  with the aid of the difference table.

### 80. Principle of Proportional Parts.

Suppose we find from table I the logarithms of the two numbers 6257 and 6258, and we want to find the logarithm of 6257.6; or that we find from table III,  $\tan 53^\circ 23'$  and  $\tan 53^\circ 24'$ , but we want to find  $\tan 53^\circ 23' 20''$ ; or similarly, from table IV, we get  $L \cos 37^\circ 42'$  and  $L \cos 37^\circ 43'$  but we want to find  $L \cos 37^\circ 42' 48''$ ; how are we to proceed?

In order to meet such cases, the 'Principle of Proportional Parts' may be used. The principle may be stated as follows:

*If the value of a quantity depending on a variable quantity  $x$  be tabulated for different values of  $x$  at regular small intervals, then in most cases, for a very small change in  $x$  (which is called the argument) the corresponding small change in the tabulated quantity (called the function of the argument) is proportional to the change in  $x$ .*

We shall assume the truth of this principle; for a strict proof of it, with the proper restriction under which it is true, depends on the use of Calculus. For the tables with which we are concerned, it is true for all practical purposes.

The application of the principle is illustrated in the following examples:

**Ex. 1.** *Given  $\log 63374 = 4.8019111$  and  $\log 63375 = 4.8019180$ , find  $\log 63374.3$  and find the number whose logarithm is  $\bar{2}.8019136$ .*

Here,  $\log 63375 = 4.8019180$

and  $\log 63374 = 4.8019111.$

Hence, for an increase of 1 in the number, the increment in the logarithm is '0000069. (This is usually spoken as 'diff. for 1 is 69')

Therefore, by the Principle of Proportional Parts, increase in the logarithm for an increase of '3 in the number is

$$'3 \times '0000069 = '00000207$$

$$= '0000021, \text{ up to seven places.}$$

$$\text{Hence, } \log 63374'3 = 4'8019111 + '0000021$$

$$= 4'8019132.$$

$$\therefore \log 63'3743 = 1'8019132.$$

Again, 4'8019136 lies between 4'8019111 and 4'8019180, the difference from the former being '0000025. Hence, 4'8019136 is the logarithm of a number lying between 63374 and 63375, say logarithm of  $63374 + x$ .

Then, diff. for 1 being 69 (i.e., '0000069) and diff. for  $x$  being 25, (i.e., '0000025), by the Principle of Proportional Parts, we have

$$69 : 25 :: 1 : x$$

$$\text{or, } x = \frac{25}{69} = '36 \dots \dots$$

$$\text{Hence, } \log 63374'36 \dots = 4'8019136.$$

The required number whose logarithm is  $\bar{2}'8019136$ , having the same mantissa, must be formed of the same digits arranged in the same order, and its characteristic being -2, the number must be '06337436...

$$\text{Ex. 2. (i) Given } L \sin 37^\circ 43' 50'' = 9'7867152$$

$$L \sin 37^\circ 44' = 9'7867424,$$

find  $L \sin 37^\circ 43' 56''$ .

$$\text{(ii) Given } L \tan 79^\circ 51' 40'' = 10'7475657$$

$$L \tan 79^\circ 51' 50'' = 10'7476872,$$

find the angle whose  $L \tan$  is 10'7476532.

In (i) diff. (in the value of  $L \sin$ ) for  $10''$  (diff. in angle)  
 $= 272$  (i.e., '0000272)

hence, diff. for  $6'' = \frac{6}{10} \times 272 = 163.2$  i.e., '00001632

and so  $L \sin 37^\circ 43' 56'' = 9.7867152 + '0000163$   
 $= 9.7867315$ .

In (ii) the angle whose  $L \tan$  is  $10.7476532$  evidently lies between  $79^\circ 51' 40''$  and  $79^\circ 51' 50''$ .

Let the angle be  $79^\circ 51' 40'' + x''$ .

Now, diff. (in the value of  $L \tan$ ) for  $10''$  (diff. in angle)  
 $= 1215$  (i.e., 0001215)

and diff. for  $x'' = 875$

(i.e., 0000875, being  $10.7476532 - 10.7475657$ )

$\therefore \frac{x}{10} = \frac{875}{1215}$  or  $x = 7.2$  nearly.

Thus, the required angle is  $79^\circ 51' 47''.2$ .

**Ex. 3.** Given  $\cos 58^\circ 17' = .5257191$  and diff. for  $1' = 2474$ ,  
 find  $\cos 58^\circ 17' 20''$ .

Here, diff. for  $1'$  i.e.,  $60'' = 2474$ ,

$\therefore$  diff. for  $20'' = \frac{20}{60} \times 2474 = 825$  (nearly).

As for increasing angle, cosine diminishes,

$\therefore \cos 58^\circ 17' 20'' = .5257191 - .0000825$   
 $= .5256366$ .

### Examples XIII(b)

1. Given  $\log 18.906 = 1.2765997$   
 and  $\log 18.907 = 1.2766226$ ,  
 find  $\log 1890.635$ .

2. Given  $\log 69714 = 4.8433200$   
 $\log 69715 = 4.8433262$ ,  
 find  $\log (.000697145)^{\frac{1}{2}}$ .

3. Given  $\log 37602 = 4.5752109$

$\log 37601 = 4.5751994,$

find the number whose logarithm is 1.5752086.

4. Given  $\log 3 = .4771213$

$\log 74008 = 4.8692787$

diff. for  $1' = 59,$

find  $(.09)^{\frac{1}{5}}.$

5. Given  $\cos 32^{\circ} 16' = .8455726$

and  $\cos 32^{\circ} 17' = .8454172,$

find the value of  $\cos 32^{\circ} 16' 24''$

and find the angle whose cosine is .8455176.

6. Find  $\tan 38^{\circ} 24' 37.5''$ , having given

$\tan 38^{\circ} 24' = .7925902$  and  $\tan 38^{\circ} 25' = .7930640.$

7. Given  $L \sin 44^{\circ} 17' = 9.8439842$

and  $L \sin 44^{\circ} 18' = 9.8441137,$

find  $L \sin 44^{\circ} 17' 33''$ . Deduce the value of

$L \operatorname{cosec} 44^{\circ} 17' 33''.$

8. Given  $L \sin 36^{\circ} 24' = 9.7733614$

$L \sin 36^{\circ} 25' = 9.7735327,$

find the angle whose  $L \sin$  is 9.7734642.

9. If  $L \cot 53^{\circ} 13' = 9.8736937$

$L \cot 53^{\circ} 14' = 9.8734302,$

find  $\theta$  where  $L \cot \theta = 9.8734523.$

10. Given  $L \tan 22^{\circ} 37' = 9.6197205$

diff. for  $1' = 3557,$

find the value of

$L \tan 22^{\circ} 37' 22''$

and the angle whose  $L \tan$  is 9.6195283.



11. Prove that,  $\theta$  being any acute angle,

$$\begin{aligned} L \sin \theta + L \operatorname{cosec} \theta &= L \cos \theta + L \sec \theta \\ &- L \tan \theta + L \cot \theta = 20. \end{aligned}$$

12. Given  $L \cos 36^\circ 40' = 9.9042411$ , find  $L \sec 36^\circ 40'$ .

13. Given  $L \cos 34^\circ 44' = 9.9147729$

$$L \cos 34^\circ 45' = 9.9146852,$$

find the value of  $L \cos 34^\circ 44' 27''$ .

14. Given  $L \sin 36^\circ 40' = 9.7760897$

$$L \cos 36^\circ 40' = 9.9042411,$$

find  $L \tan 36^\circ 40'$ .

15. Prove that the difference of tabular logarithms of any two ratios is equal to the difference of the logarithms of those two ratios.

16. If  $\sin \theta = .8$ , find  $\theta$ ,

$$\text{given } \log 2 = .3010300,$$

$$L \sin 53^\circ 7' = 9.9030136$$

$$L \sec 36^\circ 52' = 10.0968916.$$

17. Find the value of

$$\frac{\sin 34^\circ 17' \times \cos 77^\circ 23'}{\tan 27^\circ 12'}.$$

$$\text{given } L \sin 12^\circ 37' = 9.3393$$

$$L \cos 55^\circ 43' = 9.7507$$

$$L \tan 62^\circ 48' = 10.2891$$

$$\text{and } \log 23.94 = 1.3791$$


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# CHAPTER XIV

## PROPERTIES OF TRIANGLES

81. In a triangle  $ABC$ , there are six parts, the three sides and the three angles. It is usual to denote the angles of the triangle by  $A, B, C$  and the corresponding opposite sides by  $a, b, c$ . The six parts are not independent of one another. The various relations existing among them are deduced in the following articles.

82. In any triangle, prove that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

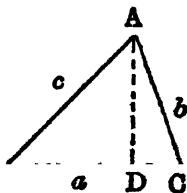


Fig. (i)

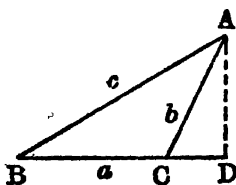


Fig. (ii)

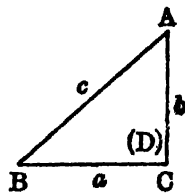


Fig. (iii)

Let  $ABC$  be any triangle. From  $A$  draw  $AD$  perpendicular to  $BC$  or  $BC$  produced if necessary [ Fig. (ii) ]

[ In Fig. (i),  $C$  is an *acute* angle, in Fig. (ii),  $C$  is an *obtuse* angle, in Fig. (iii),  $C$  is a *right* angle. ]

From  $\triangle ABD$ ,  $AD = AB \sin ABD = c \sin B$ .

From  $\triangle ACD$ ,  $AD = AC \sin ACD = b \sin C$  [ Fig. (i) ]

or,  $= b \sin (\pi - C)$  [ Fig. (ii) ]

i.e.,  $= b \sin C$ .

$$\therefore b \sin C = c \sin B, \text{ i.e., } \sin B = \sin C$$

Similarly, by drawing a perpendicular from  $B$  upon  $CA$ , we have  $\frac{a}{\sin A} = \frac{c}{\sin C}$ .

In Fig. (iii),  $C$  is a *right angle* ;

$$\sin A = \frac{a}{c} ; \sin B = \frac{b}{c} ; \sin C = 1.$$

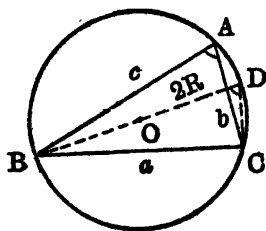
$$\therefore \frac{a}{\sin A} = \frac{b}{\sin B} = c = \frac{c}{\sin C}.$$

Hence, in all cases,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \dots \quad (1)$$

Thus, in any triangle, the sides are proportional to the sines of the opposite angles.

An alternative method of proof.



Let  $O$  be the centre and  $R$  be the radius of the circle circumscribing the triangle  $ABC$ .

Join  $BO$  and produce it to meet the circumference in  $D$ . Join  $CD$ . The  $\angle BCD$  is then a right angle.

$$\text{From } \triangle BCD, \sin BDC = \frac{BC}{BD} = \frac{a}{2R}.$$

But  $\angle BDC = \angle A$ , being in the same segment,

$$\therefore \frac{a}{2R} = \sin A, \text{ or, } \frac{a}{\sin A} = 2R.$$

Similarly, by joining  $AO$  and producing it to meet the circumference in  $E$ , and joining  $CE$ ,  $BE$ , it can be shown that

$$\begin{aligned} \frac{b}{\sin B} &= 2R \text{ and } \frac{c}{\sin C} = 2R. \\ \therefore \frac{a}{\sin A} &= \frac{b}{\sin B} = \frac{c}{\sin C} = 2R. \quad \dots (2) \end{aligned}$$

**Note 1.** If angle  $A$  be obtuse,  $A$  and  $D$  fall on opposite sides of  $BC$  and  $ABCD$  being cyclic,  $\sin BDC = \sin (180^\circ - A) = \sin A$ , and the same result follows. In case  $A$  is a right angle, evidently  $2R = a = a/\sin A$ , and we get the same result.

**Note 2.** It follows from the relation (2) that

$$\begin{aligned} a &= 2R \sin A, b = 2R \sin B, c = 2R \sin C; \\ \sin A &= \frac{a}{2R}, \sin B = \frac{b}{2R}, \sin C = \frac{c}{2R}. \end{aligned}$$

83. In any triangle, to prove that

$$a^2 = b^2 + c^2 - 2bc \cos A, \text{ or, } \cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

$$b^2 = c^2 + a^2 - 2ca \cos B, \text{ or, } \cos B = \frac{c^2 + a^2 - b^2}{2ca}.$$

$$c^2 = a^2 + b^2 - 2ab \cos C, \text{ or, } \cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

Take the figures of Art. 82.

First, let  $C$  be an *acute* angle [Fig. (i)]; then from Geometry,

$$AB^2 = BC^2 + CA^2 - 2BC \cdot CD.$$

Now, from  $\triangle ACD$ ,  $CD = AC \cos C = b \cos C$ .

$$\therefore c^2 = a^2 + b^2 - 2ab \cos C.$$

Next, let the angle  $C$  be an *obtuse* angle [Fig. (ii)]; then from Geometry,

$$AB^2 = BC^2 + CA^2 + 2BC \cdot CD.$$

Now, from  $\triangle ACD$ ,  $CD = AC \cos ACD$

$$= b \cos (\pi - C) = -b \cos C.$$

$$\therefore c^2 = a^2 + b^2 - 2ab \cos C.$$

Lastly, let  $C$  be a *right angle* [Fig. (iii)] ; then from Geometry,

$$AB^2 = BC^2 + CA^2,$$

$$\text{i.e., } c^2 = a^2 + b^2 = a^2 + b^2 - 2ab \cos C.$$

$$[\because \cos C = \cos 90^\circ = 0.]$$

Hence, for all values of  $C$ , we have

$$c^2 = a^2 + b^2 - 2ab \cos C.$$

$$\therefore \cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

Similarly, the other two relations can be established.

**Obs.** This theorem expresses the cosines of the angles of a triangle in terms of the sides.

**84.** In any triangle, to prove that

$$a = b \cos C + c \cos B.$$

$$b = c \cos A + a \cos C.$$

$$c = a \cos B + b \cos A.$$

Take the figures of Art. 82.

In Fig. (i), where  $C$  is an *acute angle*,

$$BC = BD + CD$$

$$= AB \cos ABD + AC \cos ACD.$$

$$\therefore a = c \cos B + b \cos C.$$

In Fig. (ii), where  $C$  is an *obtuse angle*,

$$BC = BD - CD$$

$$= AB \cos ABD - AC \cos ACD$$

$$= c \cos B - b \cos (180^\circ - C).$$

$$= c \cos B + b \cos C.$$

In Fig. (iii), where  $C$  is a *right angle*,

$$BC = AB \cos B.$$

$$\therefore a = c \cos B = c \cos B + b \cos C.$$

$$[\because \cos C = \cos 90^\circ = 0.]$$

Thus, in all cases,

$$a = b \cos C + c \cos B.$$

Similarly, the other two relations can be established.

85. From Art. 83 and note of Art. 82, it follows that

$$\tan A = \frac{\sin A}{\cos B} = \frac{\frac{a}{2R}}{\frac{b^2 + c^2 - a^2}{2bc}} = \frac{abc}{R(b^2 + c^2 - a^2)}$$

$$\text{Similarly, } \tan B = \frac{abc}{R(c^2 + a^2 - b^2)}.$$

$$\tan C = \frac{abc}{R(a^2 + b^2 - c^2)}.$$

86. Trigonometrical ratios of half angles of a triangle in terms of the sides.

$$\begin{aligned} \text{We have, } 2 \sin^2 \frac{A}{2} &= 1 - \cos A = 1 - \frac{b^2 + c^2 - a^2}{2bc} \\ &= \frac{2bc - b^2 - c^2 + a^2}{2bc} = \frac{a^2 - (b^2 - 2bc + c^2)}{2bc} \\ &= \frac{a^2 - (b - c)^2}{2bc} = \frac{(a - b + c)(a + b - c)}{2bc}. \end{aligned}$$

Let  $s$  denote the semi-perimeter of the triangle ;

$$\text{then } 2s = a + b + c.$$

$$\text{Now, } a - b + c = a + b + c - 2b = 2s - 2b = 2(s - b),$$

$$a + b - c = a + b + c - 2c = 2s - 2c = 2(s - c).$$

$$\text{Hence, } 2 \sin^2 \frac{A}{2} = \frac{2(s - b) \cdot 2(s - c)}{2bc}$$

$$\text{i.e., } \sin^2 \frac{A}{2} = \frac{(s - b)(s - c)}{bc}.$$

$$\therefore \sin \frac{A}{2} = \sqrt{\frac{(s - b)(s - c)}{bc}}.$$

The positive value of the square root must be taken for  $A$ , being an angle of a triangle, is less than  $180^\circ$ ; and hence,  $\frac{1}{2}A < 90^\circ$  and consequently,  $\sin \frac{1}{2}A$  must always be positive.

$$\begin{aligned}\text{Again, } 2 \cos^2 \frac{A}{2} &= 1 + \cos A \\ &= 1 + \frac{b^2 + c^2 - a^2}{2bc} = \frac{2bc + b^2 + c^2 - a^2}{2bc} \\ &= \frac{(b+c)^2 - a^2}{2bc} = \frac{(b+c+a)(b+c-a)}{2bc}.\end{aligned}$$

$$\text{Now, } b+c-a = a+b+c-2a = 2s-2a = 2(s-a).$$

$$\therefore 2 \cos^2 \frac{A}{2} = \frac{2s \cdot 2(s-a)}{2bc}, \text{ i.e., } \cos^2 \frac{A}{2} = \frac{s(s-a)}{bc}.$$

$$\therefore \cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}.$$

Here also the positive value of the square root must be taken; for  $\frac{1}{2}A$  being less than  $90^\circ$ ,  $\cos \frac{1}{2}A$  is always positive.

$$\begin{aligned}\text{Again, } \tan \frac{A}{2} &= \sin \frac{A}{2} + \cos \frac{A}{2} \\ &= \sqrt{\frac{(s-b)(s-c)}{bc}} + \sqrt{\frac{s(s-a)}{bc}} \\ &= \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.\end{aligned}$$

Similarly, the trigonometrical ratios of  $\frac{B}{2}$ ,  $\frac{C}{2}$  can be obtained in terms of the sides.

**Note.** Without assuming the values of  $\sin \frac{1}{2}A$ ,  $\cos \frac{1}{2}A$ , the value of  $\tan \frac{1}{2}A$  can be obtained by substituting the values of  $\cos A$  in terms of the sides from Art. 83 in the relation  $\tan^2 \frac{1}{2}A = \frac{1-\cos A}{1+\cos A}$  and then extracting the square root after simplification.

Thus, we have

$$\left. \begin{aligned} \sin \frac{A}{2} &= \sqrt{\frac{(s-b)(s-c)}{bc}} \\ \sin \frac{B}{2} &= \sqrt{\frac{(s-c)(s-a)}{ca}} \\ \sin \frac{C}{2} &= \sqrt{\frac{(s-a)(s-b)}{ab}} \end{aligned} \right\} \dots \dots (1)$$

$$\left. \begin{aligned} \cos \frac{A}{2} &= \sqrt{\frac{s(s-a)}{bc}} \\ \cos \frac{B}{2} &= \sqrt{\frac{s(s-b)}{ca}} \\ \cos \frac{C}{2} &= \sqrt{\frac{s(s-c)}{ab}} \end{aligned} \right\} \dots \dots (2)$$

$$\left. \begin{aligned} \tan \frac{A}{2} &= \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \\ \tan \frac{B}{2} &= \sqrt{\frac{(s-c)(s-a)}{s(s-b)}} \\ \tan \frac{C}{2} &= \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} \end{aligned} \right\} \dots \dots (3)$$

87. Sine of an angle of a triangle in terms of the sides.

$$\begin{aligned} \sin A &= 2 \sin \frac{A}{2} \cos \frac{A}{2} \\ &= 2 \sqrt{\frac{(s-b)(s-c)}{bc}} \cdot \sqrt{\frac{s(s-a)}{bc}}. \quad [\text{Art. 86}] \end{aligned}$$

$$\therefore \sin A = \frac{2}{bc} \sqrt{s(s-a)(s-b)(s-c)}.$$

$$\text{Similarly, } \sin B = \frac{2}{ca} \sqrt{s(s-a)(s-b)(s-c)}.$$

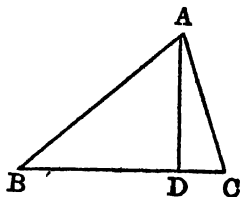
$$\sin C = \frac{2}{ab} \sqrt{s(s-a)(s-b)(s-c)}.$$



$\sqrt{s(s-a)(s-b)(s-c)}$ , being the expression for the area of the triangle [See Art. 88], is usually denoted by the Greek letter  $\Delta$ . Hence, the above formulæ may be written as

$$\sin A = \frac{2\Delta}{bc}, \sin B = \frac{2\Delta}{ca}, \sin C = \frac{2\Delta}{ab}.$$

### 88. Area of a triangle.



Let  $ABC$  be a triangle and let  $\Delta$  denote its area. Draw  $AD$  perpendicular to  $BC$ ; then from  $\triangle ACD$ ,

$$AD = AC \sin C = b \sin C.$$

$$\text{Now, } \Delta = \frac{1}{2}BC \cdot AD = \frac{1}{2}ab \sin C.$$

Similarly by drawing perpendiculars from  $B$  and  $C$  to the opposite sides, it can be shown that

$$\Delta = \frac{1}{2}bc \sin A = \frac{1}{2}ca \sin B.$$

$$\text{Otherwise, } \Delta = \frac{1}{2}ab \sin C$$

$$= \frac{1}{2}ca \sin B \quad [\because b \sin C = c \sin B]$$

$$= \frac{1}{2}bc \sin A \quad [\because a \sin B = b \sin A]$$

$$\text{Thus, } \Delta = \frac{1}{2}bc \sin A = \frac{1}{2}ca \sin B = \frac{1}{2}ab \sin C \quad \dots \quad (i)$$

$$= \frac{1}{2}(\text{product of two sides}) \times (\text{sine of included angle}).$$

$$\text{Again, } \Delta = \frac{1}{2}bc \sin A = bc \sin \frac{A}{2} \cos \frac{A}{2}$$

$$= bc \sqrt{\frac{(s-b)(s-c)}{bc}} \sqrt{\frac{s(s-a)}{bc}}$$

$$= \sqrt{s(s-a)(s-b)(s-c)}. \quad (ii)$$

Substituting in the expression  $s = \frac{1}{2}(a + b + c)$ ,  
we get

$$\begin{aligned}\Delta &= \frac{1}{4} \sqrt{(a+b+c)(b+c-a)(c+a-b)(a+b-c)} \\ &= \frac{1}{4} \{2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4\}^{\frac{1}{2}}. \quad \dots \quad (iii)\end{aligned}$$

Again,

$$\Delta = \frac{1}{2}bc \sin A = \frac{1}{2}bc \cdot \frac{a}{2R} \quad [\text{Art. 82}] = \frac{abc}{4R}. \quad \dots \quad (iv)$$

**Note.** In some text books,  $S$  is used to denote the area of a triangle, but to avoid confusion between  $S$  and  $s$  in writing, the symbol  $\Delta$  is preferable.

89. In any triangle, to prove that

$$\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2}.$$

We have, in any triangle,

$$\frac{b}{c} = \frac{\sin B}{\sin C}.$$

$$\begin{aligned}\therefore \frac{b-c}{b+c} &= \frac{\sin B - \sin C}{\sin B + \sin C} = \frac{2 \cos \frac{B+C}{2} \sin \frac{B-C}{2}}{2 \sin \frac{B+C}{2} \cos \frac{B-C}{2}} \\ &= \cot \frac{B+C}{2} \tan \frac{B-C}{2} \\ &= \tan \frac{A}{2} \tan \frac{B-C}{2} \left[ \because \frac{A}{2} + \frac{B+C}{2} = 90^\circ \right] \\ \therefore \tan \frac{B-C}{2} &= \frac{b-c}{b+c} \cdot \frac{1}{\tan \frac{A}{2}} = \frac{b-c}{b+c} \cot \frac{A}{2}.\end{aligned}$$

Similarly,

$$\tan \frac{C-A}{2} = \frac{c-a}{c+a} \cot \frac{B}{2}, \quad \tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2}.$$

90. The three sets of formulæ in Arts. 82, 83, 84 have been established directly from the figures. These three sets

however, are not independent, for, from any one set, the other two sets can be deduced.

For example, let us deduce the formulæ of Art. 83 from those of Art. 84.

$$\text{By Art. 84, } a = b \cos C + c \cos B$$

$$b = c \cos A + a \cos C$$

$$c = a \cos B + b \cos A.$$

Multiplying these in succession by  $a$ ,  $b$ , and  $c$ , and subtracting the first result from the sum of the other two, we have,

$$\begin{aligned} b^2 + c^2 - a^2 &= b(c \cos A + a \cos C) + c(a \cos B + b \cos A) \\ &\quad - a(b \cos C + c \cos B) = 2bc \cos A. \end{aligned}$$

$$\therefore \cos A = \frac{b^2 + c^2 - a^2}{2bc}; \text{ similarly, for } \cos B, \cos C.$$

*Note.* For other cases, see *Appendix*.

91. In working out identities which involve both the sides and angles of a triangle, it is sometimes convenient to express the sides in terms of the angles or the angles in terms of the sides.

Also, it is sometimes found convenient to, express the values of  $\tan \frac{A}{2}$ ,  $\tan \frac{B}{2}$ ,  $\tan \frac{C}{2}$  in a form in which the denominator is constant and numerator is free from radical. Thus, multiplying the numerator and the denominator of the value of  $\tan \frac{A}{2}$  by  $\sqrt{(s-b)(s-c)}$  and noting that

$$\begin{aligned} \sqrt{s(s-a)(s-b)(s-c)} &= \Delta, \text{ we have} \\ \tan \frac{A}{2} &= \frac{(s-b)(s-c)}{\Delta}; \text{ similarly, } \tan \frac{B}{2} = \frac{(s-c)(s-a)}{\Delta}; \\ \tan \frac{C}{2} &= \frac{(s-a)(s-b)}{\Delta}. \end{aligned}$$

Again, multiplying the numerator and the denominator of the value of  $\cot \frac{A}{2}$  by  $\sqrt{s(s-a)}$  we have

$$\cot \frac{A}{2} = \frac{s(s-a)}{\Delta}.$$

$$\text{Similarly, } \cot \frac{B}{2} = \frac{s(s-b)}{\Delta}; \cot \frac{C}{2} = \frac{s(s-c)}{\Delta}.$$

**Ex. 1.** Show that in any triangle,

$$a(\sin B - \sin C) + b(\sin C - \sin A) + c(\sin A - \sin B) = 0.$$

$$\begin{aligned} \text{Left side} &= (a \sin B - b \sin A) + (b \sin C - c \sin B) \\ &\quad + (c \sin A - a \sin C) \end{aligned}$$

$$\begin{aligned} &= 0 + 0 + 0 \left[ \because \text{by Art. 82, } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \right] \\ &= 0. \end{aligned}$$

**Ex. 2.** Show that in any triangle,

$$a \sin (B - C) + b \sin (C - A) + c \sin (A - B) = 0.$$

[ H. S. 1961 ]

$$\begin{aligned} a &= 2R \sin A \text{ [ by Art. 82 ]} = 2R \sin (B + C), \\ &\quad \text{[ } \because A + B + C = \pi \text{ ]} \end{aligned}$$

$$\begin{aligned} \therefore \text{ 1st term of the left side} &= 2R \sin (B + C) \sin (B - C) \\ &= 2R (\sin^2 B - \sin^2 C). \end{aligned}$$

[ by Ex. 2, Art. 35 ]

$$\begin{aligned} \text{Similarly, 2nd term} &= 2R (\sin^2 C - \sin^2 A) \\ \text{3rd term} &= 2R (\sin^2 A - \sin^2 B). \end{aligned}$$

Now adding together the three terms, the required result follows.

**Ex. 3.** In any triangle, prove that

$$(b - c) \cot \frac{A}{2} + (c - a) \cot \frac{B}{2} + (a - b) \cot \frac{C}{2} = 0.$$

Substituting the values of  $\cot \frac{A}{2}$ ,  $\cot \frac{B}{2}$ ,  $\cot \frac{C}{2}$ , as given in Art. 91, we have, the left side

$$\begin{aligned} &= (b-c) \cdot \frac{s(s-a)}{\Delta} + (c-a) \cdot \frac{s(s-b)}{\Delta} + (a-b) \cdot \frac{s(s-c)}{\Delta} \\ &= \frac{s}{\Delta} \left[ (b-c)(s-a) + (c-a)(s-b) + (a-b)(s-c) \right] \\ &= \frac{s}{\Delta} \cdot 0 = 0. \end{aligned}$$

**Ex. 4.** *If the cosines of two of the angles of a triangle are inversely proportional to the opposite sides, show that the triangle is either isosceles or right-angled.*

We have, by the question,

$$\frac{\cos A}{\cos B} = \frac{b}{a} = \frac{\sin B}{\sin A} \quad [\text{by Art. 82}]$$

$$\therefore \sin A \cos A = \sin B \cos B, \text{ or, } \sin 2A = \sin 2B,$$

$$\text{or, } \sin 2A - \sin 2B = 0,$$

$$\text{or, } 2 \cos (A+B) \sin (A-B) = 0.$$

$$\therefore \text{either } \cos (A+B) = 0, \text{ i.e., } (A+B) = 90^\circ,$$

i.e., the triangle is right-angled ;

$$\text{or, } \sin (A-B) = 0, \text{ i.e., } A-B=0, \text{ i.e., } A=B,$$

i.e., the triangle is isosceles.

**Ex. 5.** *If the sides of a triangle are in A.P., show that  $\cot \frac{A}{2}$ ,  $\cot \frac{B}{2}$ ,  $\cot \frac{C}{2}$  are also in A.P.*

$$\cot \frac{A}{2}, \cot \frac{B}{2}, \cot \frac{C}{2} \text{ are in A.P.,}$$

$$\text{if } \cot \frac{B}{2} - \cot \frac{A}{2} = \cot \frac{C}{2} - \cot \frac{B}{2}.$$

$$\text{i.e., if } \frac{s(s-b)}{\Delta} - \frac{s(s-a)}{\Delta} = \frac{s(s-c)}{\Delta} - \frac{s(s-b)}{\Delta},$$

$$\text{i.e., if } (s-b) - (s-a) = (s-c) - (s-b),$$

$$\text{i.e., if } a - b = b - c,$$

$$\text{i.e., if } a, b, c \text{ are in A.P.}$$

**Ex. 6.** Show that

$$b^2 \sin 2C + c^2 \sin 2B = 4\Delta.$$

$$\text{Left side} = b^2 \cdot 2 \sin C \cos C + c^2 \cdot 2 \sin B \cos B$$

$$= 2b \sin C \cdot b \cos C + 2c \sin B \cdot c \cos B$$

$$= 2b \sin C (b \cos C + c \cos B)$$

$$[ \because c \sin B = b \sin C ]$$

$$= 2ab \sin C \quad [ \text{by Art. 84} ]$$

$$= 4 \cdot \frac{1}{2} ab \sin C = 4\Delta. \quad [ \text{by Art. 88} ]$$

### Examples XIV(a)

In any triangle, prove that (*Ex. 1 to 21*):—

$$1. \quad \sin \frac{B-C}{2} = \frac{b-c}{a} \cos \frac{A}{2} \quad [ \text{H. S. 1961 Comp.} ]$$

$$2. \quad \cos \frac{B-C}{2} = \frac{b+c}{a} \sin \frac{A}{2} \quad [ \text{H. S. 1961 Comp.} ]$$

$$3. \quad (b+c) \cos A + (c+a) \cos B + (a+b) \cos C = a+b+c.$$

$$4. \quad \frac{a+b}{a-b} = \tan \frac{A+B}{2} \cot \frac{A-B}{2}.$$

$$5. \quad a^2 + b^2 + c^2 = 2(bc \cos A + ca \cos B + ab \cos C).$$

$$6. \quad (b+c-a) \tan \frac{A}{2} = (c+a-b) \tan \frac{B}{2} = (a+b-c) \tan \frac{C}{2}.$$

7.  $\frac{a \sin (B-C)}{b^2-c^2} = \frac{b \sin (C-A)}{c^2-a^2} = \frac{c \sin (A-B)}{a^2-b^2}.$
8.  $a^2 (\sin^2 B - \sin^2 C) + b^2 (\sin^2 C - \sin^2 A) + c^2 (\sin^2 A - \sin^2 B) = 0.$
9.  $a^2 (\cos^2 B - \cos^2 C) + b^2 (\cos^2 C - \cos^2 A) + c^2 (\cos^2 A - \cos^2 B) = 0.$
10.  $\frac{a^2 \sin (B-C)}{\sin B + \sin C} + \frac{b^2 \sin (C-A)}{\sin C + \sin A} + \frac{c^2 \sin (A-B)}{\sin A + \sin B} = 0.$
11.  $a \sin \frac{A}{2} \sin \frac{B-C}{2} + b \sin \frac{B}{2} \sin \frac{C-A}{2} + c \sin \frac{C}{2} \sin \frac{A-B}{2} = 0.$
12.  $\frac{b^2-c^2}{a^2} \sin 2A + \frac{c^2-a^2}{b^2} \sin 2B + \frac{a^2-b^2}{c^2} \sin 2C = 0.$
13.  $a^3 \sin (B-C) + b^3 \sin (C-A) + c^3 \sin (A-B) = 0.$
14.  $a^3 \cos (B-C) + b^3 \cos (C-A) + c^3 \cos (A-B) = 3abc.$
15.  $\frac{a^2 \sin (B-C)}{\sin A} + \frac{b^2 \sin (C-A)}{\sin B} + \frac{c^2 \sin (A-B)}{\sin C} = 0.$
16.  $(b^2-c^2) \cot A + (c^2-a^2) \cot B + (a^2-b^2) \cot C = 0.$
17.  $\frac{b^2-c^2}{\cos B + \cos C} + \frac{c^2-a^2}{\cos C + \cos A} + \frac{a^2-b^2}{\cos A + \cos B} = 0.$
18.  $(s-a) \tan \frac{A}{2} = (s-b) \tan \frac{B}{2} = (s-c) \tan \frac{C}{2}.$
19.  $\frac{b-c}{a} \cos^2 \frac{A}{2} + \frac{c-a}{b} \cos^2 \frac{B}{2} + \frac{a-b}{c} \cos^2 \frac{C}{2} = 0.$
20.  $bc \cos^2 \frac{A}{2} + ca \cos^2 \frac{B}{2} + ab \cos^2 \frac{C}{2} = s^2.$
21.  $\frac{1}{a} \cos^2 \frac{A}{2} + \frac{1}{b} \cos^2 \frac{B}{2} + \frac{1}{c} \cos^2 \frac{C}{2} = \frac{s^2}{abc}.$
22. If  $A$  be  $60^\circ$ , show that  $b+c=2a \cos \frac{B-C}{2}.$

23. Show that a triangle having its sides equal to 3, 5, 7 is an obtuse-angled triangle and determine the obtuse angle.

24. Given  $(a+b+c)(b+c-a) = 3bc$ , find  $A$ .

25. If  $c^4 - 2(a^2 + b^2)c^2 + a^4 + a^2b^2 + b^4 = 0$ , prove that  $C = 60^\circ$ , or,  $120^\circ$ .

26. If  $a^4 + b^4 + c^4 = 2c^2(a^2 + b^2)$ , prove that  $C = 45^\circ$ , or,  $135^\circ$ .

27. The sides of a triangle are  $2x+3$ ,  $x^2+3x+3$ ,  $x^2+2x$ ; show that the greatest angle is  $120^\circ$ .

28. If  $\frac{1}{a+c} + \frac{1}{b+c} = \frac{3}{a+b+c}$ , show that  $C = 60^\circ$ .

29. If  $a = 2b$  and  $A = 3B$ , find the angles of the triangle.

30. If the cosines of two of the angles of a triangle are proportional to the opposite sides, show that the triangle is isosceles.

31. If  $\cos A = \frac{\sin B}{2 \sin C}$ , show that the triangle is isosceles.

32. If  $(a^2 + b^2) \sin(A-B) = (a^2 - b^2) \sin(A+B)$ , prove that the triangle is either isosceles or right-angled.

✓33. If  $(\cos A + 2 \cos C) : (\cos A + 2 \cos B) = \sin B : \sin C$ , prove that the triangle is either isosceles or right-angled.

34. If  $a^2, b^2, c^2$  be in A.P., prove that  $\cot A, \cot B, \cot C$  are also in A.P.

35. If  $a \cos^2 \frac{C}{2} + c \cos^2 \frac{A}{2} = \frac{3b}{2}$ , show that the sides of the triangle are in A.P.

36. If  $\sin A : \sin C = \sin(A-B) : \sin(B-C)$ , show that  $a^2, b^2, c^2$  are in A.P.



37. If  $a, b, c$  are in A.P., show that

$\cos A \cot \frac{1}{2}A, \cos B \cot \frac{1}{2}B, \cos C \cot \frac{1}{2}C$  are in A.P.

$$[ \cos A \cot \frac{1}{2}A = (1 - 2 \sin^2 \frac{1}{2}A) \cot \frac{1}{2}A = \cot \frac{1}{2}A - \sin A. ]$$

38. Assuming  $\Delta = \frac{1}{2}bc \sin A$  and using the value of  $\cos A$  in terms of sides, show that

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}.$$

39. Find the area of the triangle whose sides are

$$\frac{y}{z} + \frac{z}{x}, \frac{z}{x} + \frac{x}{y}, \frac{x}{y} + \frac{y}{z}.$$

40. In a triangle, if  $a = 13, b = 14, c = 15$ , find its area.

Prove that in any triangle :

$$41. \frac{a^2 - b^2}{2} \cdot \frac{\sin A \sin B}{\sin(A-B)} = \Delta.$$

$$42. 4\Delta (\cot A + \cot B + \cot C) = a^2 + b^2 + c^2.$$

$$43. a \cos A + b \cos B + c \cos C = 4R \sin A \sin B \sin C.$$

$$44. a \sin B \sin C + b \sin C \sin A + c \sin A \sin B = \frac{3\Delta}{R}.$$

$$45. (a \sin A + b \sin B + c \sin C)^2 \\ = (a^2 + b^2 + c^2)(\sin^2 A + \sin^2 B + \sin^2 C).$$

$$46. \frac{\cos B \cos C}{bc} + \frac{\cos C \cos A}{ca} + \frac{\cos A \cos B}{ab} = \frac{1}{4R^2}.$$

[ Use  $\Sigma \cot B \cot C = 1$ ; ex. 2, Ex. X. ]

$$47. \frac{b^2 - c^2}{a} \cos A + \frac{c^2 - a^2}{b} \cos B + \frac{a^2 - b^2}{c} \cos C = 0.$$

$$48. \frac{\cos A}{a} + \frac{a}{bc} = \frac{\cos B}{b} + \frac{b}{ca} = \frac{\cos C}{c} + \frac{c}{ab}.$$

$$49. 4\Delta = a^2 \cot A + b^2 \cot B + c^2 \cot C.$$

$$50. \left( \frac{a^2}{\sin A} + \frac{b^2}{\sin B} + \frac{c^2}{\sin C} \right) \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = \Delta.$$

### 92. Circum-radius of a triangle.

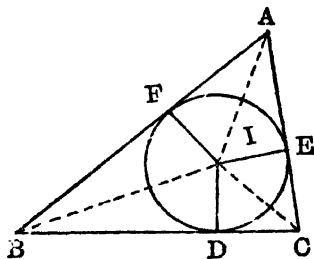
From Art. 82, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R. \quad \dots (i)$$

$$\text{Hence, } R = \frac{a}{2 \sin A} = \frac{abc}{2bc \sin A} = \frac{abc}{4 \Delta}. \quad \dots (ii)$$

### 93. In-radius of a triangle.

Let  $I$  be the centre and  $r$  the radius of the circle inscribed in the triangle  $ABC$ ; let  $D, E, F$  be the points of contact of the in-circle with the sides  $BC, CA, AB$  respectively.



Then,  $ID = IE = IF = r$ .

Join  $IA, IB, IC$ .

$$\begin{aligned} \Delta ABC &= \Delta IBC + \Delta ICA + \Delta IAB \\ &= \frac{1}{2}BC \cdot ID + \frac{1}{2}CA \cdot IE + \frac{1}{2}AB \cdot IF \\ &= \frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr \\ &= \frac{1}{2}r(a + b + c) = rs. \end{aligned}$$

Thus,  $\Delta = rs$ .

$$\therefore r = \frac{\Delta}{s}. \quad \dots (i)$$

Again,  $a = BC = BD + DC$

$$= r \cot \frac{1}{2}B + r \cot \frac{1}{2}C, \quad \text{from } \Delta^s IBD, ICD,$$

$$= r \left[ \frac{\cot \frac{1}{2}B}{\sin \frac{1}{2}B} + \frac{\cot \frac{1}{2}C}{\sin \frac{1}{2}C} \right]$$

$$= r \left[ \frac{\cos \frac{1}{2}B \sin \frac{1}{2}C + \sin \frac{1}{2}B \cos \frac{1}{2}C}{\sin \frac{1}{2}B \sin \frac{1}{2}C} \right]$$

$$= r \frac{\sin (\frac{1}{2}B + \frac{1}{2}C)}{\sin \frac{1}{2}B \sin \frac{1}{2}C} = r \frac{\cos \frac{1}{2}A}{\sin \frac{1}{2}B \sin \frac{1}{2}C}.$$

$$[ \because \frac{1}{2}A + \frac{1}{2}B + \frac{1}{2}C = 90^\circ, \sin (\frac{1}{2}B + \frac{1}{2}C) = \sin (90^\circ - \frac{1}{2}A) = \cos \frac{1}{2}A. ]$$

$$a \sin \frac{1}{2}B \sin \frac{1}{2}C \sec \frac{1}{2}A = a \frac{\sin \frac{1}{2}B \sin \frac{1}{2}C}{\cos \frac{1}{2}A}.$$

Since, by Art. 92(i),  $a = 2R \sin A = 4R \sin \frac{1}{2}A \cos \frac{1}{2}A$ ,

$$\therefore r = 4R \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C. \quad \dots (ii)$$

Since, from the figure,  $AF = AE$ ,  $BD = BF$ ,  $CD = CE$  and since the sum of these six quantities is equal to the perimeter,

$$\therefore AF + BD + CD = \text{semi-perimeter} = s.$$

$$\text{i.e., } AF + BC, \text{ or, } AF + a = s.$$

$$\therefore AF = s - a = AE.$$

Similarly,  $BF = s - b = BD$ ;  $CE = s - c = CD$ .

From  $\triangle AIF$ ,  $IF = AF \tan IAF$ .

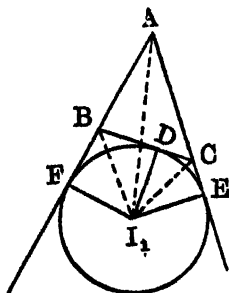
$$\left. \begin{aligned} \therefore r &= (s - a) \tan \frac{1}{2}A. \\ \text{Similarly, } r &= (s - b) \tan \frac{1}{2}B. \\ \text{and } r &= (s - c) \tan \frac{1}{2}C. \end{aligned} \right\} \quad \dots (iii)$$

**Note.** Distances of the in-centre from the vertices.

From  $\triangle AIF$ ,  $IA = IF \operatorname{cosec} IAF$ .  $\therefore IA = r \operatorname{cosec} \frac{1}{2}A$ .

Similarly,  $IB = r \operatorname{cosec} \frac{1}{2}B$  and  $IC = r \operatorname{cosec} \frac{1}{2}C$ .

#### 94. Ex-radii of a triangle.



Let  $I_1$  be the centre and  $r_1$  the radius of the escribed circle (opposite to the angle  $A$ ) of the  $\triangle ABC$ ; let  $D, E, F$  be the points of contact of the circle with the sides  $BC$ , and  $AC$  and  $AB$  produced.

Let  $r_2, r_3$  denote the radii of the escribed circles opposite to the angles  $B$  and  $C$  respectively.

Now,  $I_1D = I_1E = I_1F = r_1$  ; join  $AI_1, BI_1, CI_1$ .

$$\begin{aligned}\Delta ABC &= \Delta I_1AB + \Delta I_1AC - \Delta I_1BC \\ &= \frac{1}{2}AB \cdot I_1F + \frac{1}{2}AC \cdot I_1E - \frac{1}{2}BC \cdot I_1D \\ &= \frac{1}{2}cr_1 + \frac{1}{2}br_1 - \frac{1}{2}ar_1 \\ &= \frac{1}{2}r_1 (b + c - a) = \frac{1}{2}r_1 (b + c + a - 2a) \\ &= \frac{1}{2}r_1 (2s - 2a) \\ &= r_1 (s - a).\end{aligned}$$

Thus,  $\Delta = r_1 (s - a)$ .

$$\therefore r_1 = \frac{\Delta}{s - a}$$

Similarly,  $r_2 = \frac{\Delta}{s - b}$  (i)

$$\text{and } r_3 = \frac{\Delta}{s - c}.$$

Again,  $a = BC = BD + CD$

$$\begin{aligned}&= r_1 \cot I_1BD + r_1 \cot I_1CD, \\ &\quad \text{from } \Delta^s I_1BD, I_1CD \\ &= r_1 \cot (90^\circ - \frac{1}{2}B) + r_1 \cot (90^\circ - \frac{1}{2}C),\end{aligned}$$

because,  $\angle I_1BD = \frac{1}{2}(180^\circ - B) = 90^\circ - \frac{1}{2}B$ ,

and  $\angle I_1CD = \frac{1}{2}(180^\circ - C) = 90^\circ - \frac{1}{2}C$ .

$$\begin{aligned}\therefore a &= r_1 (\tan \frac{1}{2}B + \tan \frac{1}{2}C) \\ &= r_1 \left[ \frac{\sin \frac{1}{2}B}{\cos \frac{1}{2}B} + \frac{\sin \frac{1}{2}C}{\cos \frac{1}{2}C} \right] \\ &= r_1 \left[ \frac{\sin \frac{1}{2}B \cos \frac{1}{2}C + \sin \frac{1}{2}C \cos \frac{1}{2}B}{\cos \frac{1}{2}B \cos \frac{1}{2}C} \right] \\ &= r_1 \frac{\sin (\frac{1}{2}B + \frac{1}{2}C)}{\cos \frac{1}{2}B \cos \frac{1}{2}C} \\ &= r_1 \frac{\cos \frac{1}{2}A}{\cos \frac{1}{2}B \cos \frac{1}{2}C}, \text{ as in Art. 93.}\end{aligned}$$

$$r_1 = a \cos \frac{1}{2}B \cos \frac{1}{2}C \sec \frac{1}{2}A.$$

Putting  $a = 2R \sin A = 4R \sin \frac{1}{2}A \cos \frac{1}{2}A$ ,

$$r_1 = 4R \sin \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C.$$

Similarly,  $r_2 = 4R \cos \frac{1}{2}A \sin \frac{1}{2}B \cos \frac{1}{2}C$ , (ii)

$$\text{and } r_3 = 4R \cos \frac{1}{2}A \cos \frac{1}{2}B \sin \frac{1}{2}C.$$

Again,  $AE = AC + CE = b + CD$  [  $\because CE = CD$  ]

$$\text{and } AF = AB + BF = c + BD. \quad [ \because BF = BD ]$$

But  $AE = AF$ ; therefore, by addition, we get

$$2AE = b + c + BD + CD = b + c + a = 2s.$$

$$\therefore AE = s.$$

Again, from  $\triangle AI_1E$ ,  $I_1E = AE \tan I_1AE$ .

$$\left. \begin{array}{l} \therefore r_1 = s \tan \frac{1}{2}A. \\ \text{Similarly, } r_2 = s \tan \frac{1}{2}B, \\ \text{and } r_3 = s \tan \frac{1}{2}C. \end{array} \right\} \quad \dots \quad \dots \quad \text{(iii)}$$

**Note.** *Distances of ex-centres from the vertices.*

From  $\triangle AI_1F$ ,  $I_1A = I_1F \operatorname{cosec} I_1AF$ .

$$\begin{aligned} \therefore I_1A &= r_1 \operatorname{cosec} \frac{1}{2}A \\ &= 4R \cos \frac{1}{2}B \cos \frac{1}{2}C. \quad [ \text{by formula (ii)} ] \end{aligned}$$

From  $\triangle BI_1F$ ,  $I_1B = I_1F \operatorname{cosec} I_1BF$ .

$$\therefore I_1B = r_1 \sec \frac{1}{2}B \quad [ \because \angle I_1BF = 90^\circ - \frac{1}{2}B ]$$

Similarly,  $I_1C = r_1 \sec \frac{1}{2}C$ .

In the same way,  $I_2B = r_2 \operatorname{cosec} \frac{1}{2}B$ ,  $I_2C = r_2 \operatorname{cosec} \frac{1}{2}C$ .

**Ex. 1.** Prove that  $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r}$ .

By formula (i), Art. 94,

$$\begin{aligned} \text{left side} &= \frac{s-a}{\Delta} + \frac{s-b}{\Delta} + \frac{s-c}{\Delta} \\ &= \frac{3s - (a+b+c)}{\Delta} = \frac{3s - 2s}{\Delta} = \frac{s}{\Delta} = \frac{1}{r}. \end{aligned}$$

**Ex. 2.** Prove that  $4 \cos \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C = \frac{s}{R}$ .

$$\begin{aligned} \text{Left side} &= 4 \cdot \sqrt{\frac{s(s-a)}{bc}} \cdot \sqrt{\frac{s(s-b)}{ca}} \cdot \sqrt{\frac{s(s-c)}{ab}} \\ &= \frac{4s}{abc} \cdot \sqrt{s(s-a)(s-b)(s-c)} \\ &= \frac{4s}{abc} \Delta = s \cdot \frac{4\Delta}{abc} = \frac{s}{R} \text{ by formula (ii), Art. 92.} \end{aligned}$$

**Ex. 3.** Show that

$$\begin{aligned} \frac{bc - r_2 r_3}{r_1} &= \frac{ca - r_3 r_1}{r_2} = \frac{ab - r_1 r_2}{r_3}, \\ r_2 r_3 &= \frac{\Delta^2}{(s-b)(s-c)} = s(s-a), \\ \therefore bc - r_2 r_3 &= \frac{1}{2} [4bc - 2s(2s-2a)] \\ &= \frac{1}{2} [4bc - (a+b+c)(b+c-a)] \\ &= \frac{1}{2} [4bc + a^2 - (b+c)^2] = \frac{1}{2} [a^2 - (b-c)^2] \\ &= \frac{1}{2} [(a+b-c)(a-b+c)] = (s-b)(s-c). \\ \therefore \frac{bc - r_2 r_3}{r_1} &= \frac{(s-b)(s-c)}{r_1} = \frac{(s-a)(s-b)(s-c)}{\Delta} \\ &= \frac{\Delta}{s} = r. \end{aligned}$$

Similarly, the other ratios are equal to the same quantity.

**Ex. 4.** Prove that in any triangle,

$$r_1 + r_2 + r_3 - r = 4R.$$

$$\begin{aligned} \text{Left side} &= \left( \frac{\Delta}{s-a} + \frac{\Delta}{s-b} \right) + \left( \frac{\Delta}{s-c} - \frac{\Delta}{s} \right) \\ &= \Delta \cdot \frac{2s - (a+b)}{(s-a)(s-b)} + \Delta \cdot \frac{c}{s(s-c)} \\ &= \Delta c \left[ \frac{1}{(s-a)(s-b)} + \frac{1}{s(s-c)} \right] \quad [\because 2s = a+b+c.] \\ &= \Delta c \left[ \frac{s(s-c) + (s-a)(s-b)}{s(s-a)(s-b)(s-c)} \right] \end{aligned}$$

$$\begin{aligned}\text{Now, Numerator} &= 2s^2 - s(a+b+c) + ab \\ &= 2s^2 - s \cdot 2s + ab = ab.\end{aligned}$$

$$\text{Denominator} = \Delta^2.$$

$$\therefore \text{left side} = \frac{abc}{\Delta} = 4R.$$

**Ex. 5.** If  $r_1 = r_2 + r_3 + r$ , prove that the triangle is right-angled.

From the given relation, we have

$$r_1 - r = r_2 + r_3,$$

$$\text{or, } \frac{\Delta}{s-a} - \frac{\Delta}{s} = \frac{\Delta}{s-b} + \frac{\Delta}{s-c},$$

$$\text{or, } \frac{\Delta \cdot a}{s(s-a)} = \frac{\Delta(2s-b-c)}{(s-b)(s-c)} = \frac{\Delta \cdot a}{(s-b)(s-c)}.$$

$$\therefore s(s-a) = (s-b)(s-c).$$

$$\therefore \tan^2 \frac{1}{2}A = \frac{(s-b)(s-c)}{s(s-a)} = 1. \quad \therefore \tan \frac{1}{2}A = 1.$$

$$\therefore \frac{1}{2}A = 45^\circ.$$

$$\therefore A = 90^\circ.$$

**Note.** Although we get  $\tan \frac{1}{2}A = \pm 1$ , we reject the negative value because  $\frac{1}{2}A$  is an acute angle.

### Examples XIV(b)

Prove that in any triangle (Ex. 1 to 14) :—

$$1. \quad \sin A + \sin B + \sin C = \frac{s}{R}.$$

$$2. \quad \cos A + \cos B + \cos C = 1 + \frac{r}{R}.$$

[ Use  $\cos A + \cos B + \cos C = 1 + 4 \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C$ . ]

$$3. \quad \frac{b-c}{r_1} + \frac{c-a}{r_2} + \frac{a-b}{r_3} = 0.$$

$$4. \quad r_2 r_3 + r_3 r_1 + r_1 r_2 = s^2.$$

$$5. \quad r = R (\cos A + \cos B + \cos C - 1).$$

$$6. \quad r_1 = R (\cos B + \cos C - \cos A + 1).$$

$$[ \text{Use } \cos B + \cos C - \cos A = -1 + 4 \sin \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C ]$$

$$7. \quad a \cos B \cos C + b \cos C \cos A + c \cos A \cos B = \frac{\Delta}{R}.$$

$$8. \quad a \cot A + b \cot B + c \cot C = 2 (R + r).$$

$$\left[ a \cot A = \frac{a}{\sin A} \cdot \cos A = 2R \cos A. \quad \text{Then use Ex. 2.} \right]$$

$$9. \quad R = \frac{1}{4} \frac{(r_2 + r_3)(r_3 + r_1)(r_1 + r_2)}{r_2 r_3 + r_3 r_1 + r_1 r_2}.$$

$$10. \quad \Delta = \sqrt{rr_1 r_2 r_3} = r^2 \cot \frac{1}{2}A \cot \frac{1}{2}B \cot \frac{1}{2}C.$$

$$11. \quad \left( \frac{1}{r} - \frac{1}{r_1} \right) \left( \frac{1}{r} - \frac{1}{r_2} \right) \left( \frac{1}{r} - \frac{1}{r_3} \right) = \frac{4R}{r^2 s^2} = \frac{16R}{r^2 (a+b+c)^2}.$$

[ A. I. 1938 ]

$$12. \quad \left( \frac{1}{r} + \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \right)^2 = \frac{4}{r} \left( \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \right).$$

$$13. \quad r_1 (r_2 + r_3) \operatorname{cosec} A = r_2 (r_3 + r_1) \operatorname{cosec} B \\ = r_3 (r_1 + r_2) \operatorname{cosec} C.$$

$$14. \quad \frac{bc}{r_1} + \frac{ca}{r_2} + \frac{ab}{r_3} = 2R \left\{ \frac{b}{a} + \frac{c}{a} + \frac{c}{b} + \frac{a}{b} + \frac{a}{c} + \frac{b}{c} - 3 \right\}.$$

$$15. \quad \text{In a triangle, } a = 13, b = 14, c = 15; \text{ find } r \text{ and } R.$$

$$16. \quad \text{If } a, b, c \text{ are in A.P., show that } r_1, r_2, r_3 \text{ are in H.P.}$$

$$17. \quad \text{If in a triangle, } 3R = 4r, \text{ show that}$$

$$4 (\cos A + \cos B + \cos C) = 7.$$

$$18. \quad \text{If the diameter of an ex-circle be equal to the perimeter of the triangle, show that the triangle is right-angled.}$$

$$[ \text{Use } r_1 = s \tan \frac{1}{2}A. ]$$



19. If  $\left(1 - \frac{r_1}{r_2}\right)\left(1 - \frac{r_1}{r_3}\right) = 2$ , show that the triangle must be right-angled.

20. If  $8R^2 = a^2 + b^2 + c^2$ , show that the triangle is right-angled.

21. If  $S$  be the area of the in-circle and  $S_1, S_2, S_3$  are the areas of the escribed circles, then

$$\frac{1}{\sqrt{S}} = \frac{1}{\sqrt{S_1}} + \frac{1}{\sqrt{S_2}} + \frac{1}{\sqrt{S_3}}.$$

22. In any triangle, prove that the area of the in-circle is to the area of the triangle as  $\pi : \cot \frac{1}{2}A \cot \frac{1}{2}B \cot \frac{1}{2}C$ .

23. If  $p_1, p_2, p_3$  are the perpendiculars from the angular points of a triangle to the opposite sides, show that

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}.$$

24. If  $x, y, z$  be the lengths of the perpendiculars from the circum-centre on the sides  $BC, CA, AB$  of the triangle  $ABC$ , prove that

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = \frac{abc}{4xyz}.$$

25. If  $x, y, z$  are respectively equal to  $IA, IB, IC$ , and  $\alpha, \beta, \gamma$  are respectively equal to  $I_1A, I_2B, I_3C$ , show that

$$\text{i) } \frac{xyz}{abc} = \frac{r}{s}. \quad \text{(ii) } \frac{x}{a} + \frac{y}{\beta} + \frac{z}{\gamma} = 1.$$

$$\text{(iii) } \frac{bc}{a^2} + \frac{ca}{\beta^2} + \frac{ab}{\gamma^2} = 1. \quad \text{(iv) } ax^2 + by^2 + cz^2 = abc.$$

[ Use Notes of Arts. 93 and 94. ]

26. Prove that

$$(r_1 - r)(r_2 - r)(r_3 - r) = 4Rr^2$$

27. If  $\frac{r}{2} = \frac{r_1}{12} = \frac{R}{5}$  prove that the  $\triangle$  is right angled.

28. If  $I$  be the in-centre and  $x, y, z$  be the circumradii of the triangles  $BIC, CIA, AIB$ , then show that

$$xyz = 2R^2r.$$

29. If  $I$  be the in-centre of the triangle  $ABC$ , show that

$$IA \cdot IB \cdot IC = \frac{abc}{s} \Delta$$

where  $\Delta$  denotes the area of  $\triangle ABC$  and  $s$  is its semi-perimeter.

30. If  $D, E, F$  be the points of contact of the in-circle with the sides  $BC, CA, AB$  respectively, show that

$$EF = 2(s - a) \sin \frac{1}{2}A,$$

$$FD = 2(s - b) \sin \frac{1}{2}B,$$

$$DE = 2(s - c) \sin \frac{1}{2}C.$$

31. If the lengths of the internal bisectors of the angle of a triangle be equal, show that the triangle is isosceles.

32. If  $x, y, z$  be the lengths of the internal bisectors of the angles of a triangle and  $l, m, n$  be the lengths of those bisectors, produced to meet the circle, show that

$$(i) \frac{\cos \frac{1}{2}A}{x} + \frac{\cos \frac{1}{2}B}{y} + \frac{\cos \frac{1}{2}C}{z} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

$$(ii) l \cos \frac{1}{2}A + m \cos \frac{1}{2}B + n \cos \frac{1}{2}C = a + b + c.$$

33. If the internal bisectors of the angles of a triangle make angles  $\alpha, \beta$  and  $\gamma$  with the sides  $BC, CA$  and  $AB$  respectively, show that

$$a \sin 2\alpha + b \sin 2\beta + c \sin 2\gamma = 0.$$

34. If  $l, m, n$  be the lengths of the medians of a triangle  $ABC$ , prove that

$$(i) (b^2 - c^2)l^2 + (c^2 - a^2)m^2 + (a^2 - b^2)n^2 = 0.$$

$$(ii) 4(l^2 + m^2 + n^2) = 3(a^2 + b^2 + c^2).$$

**35.** Perpendiculars  $AL$ ,  $BM$ ,  $CN$  are drawn from the vertices  $A$ ,  $B$ ,  $C$  of an acute-angled triangle on the opposite sides and produced to meet the circumcircle of  $\triangle ABC$  in  $L'$ ,  $M'$ ,  $N'$ . If  $LL'$ ,  $MM'$ ,  $NN'$  be equal to  $\alpha$ ,  $\beta$ ,  $\gamma$  respectively, show that

$$(i) \quad \frac{\alpha}{a} + \frac{\beta}{b} + \frac{\gamma}{c} = 2(\tan A + \tan B + \tan C).$$

$$(ii) \quad \frac{AL'}{AL} + \frac{BM'}{BM} + \frac{CN'}{CN} = 4.$$

**36.** If  $x$ ,  $y$ ,  $z$  are respectively equal to  $IA$ ,  $IB$ ,  $IC$  and  $x_1$ ,  $y_1$ ,  $z_1$  are respectively equal to  $I_1A$ ,  $I_2B$ ,  $I_3C$  where  $I$  is the in-centre and  $I_1$ ,  $I_2$ ,  $I_3$  are then ex-centres of  $\triangle ABC$ , then show that

$$(i) \quad \frac{b-c}{ax_1^2} + \frac{c-a}{by_1^2} + \frac{a-b}{cz_1^2} = 0.$$

$$(ii) \quad x^2 \left( \frac{1}{b} - \frac{1}{c} \right) + y^2 \left( \frac{1}{c} - \frac{1}{a} \right) + z^2 \left( \frac{1}{a} - \frac{1}{b} \right) = 0.$$

**37.** Prove that in the triangle  $ABC$

$$\frac{IA}{I_1A} + \frac{IB}{I_2B} + \frac{IC}{I_3C} = 1.$$

**38.** Prove that in the triangle  $ABC$

$$a.BP.CP + b.CP.AP + c.AP.BP = abc$$

where  $P$  is the orthocentre of the triangle

**39.** If  $AL$ ,  $BM$ ,  $CN$  are the perpendiculars from the angular points of  $\triangle ABC$  to the opposite sides, show that

$$\frac{1}{AL} + \frac{1}{BM} + \frac{1}{CN} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}.$$

**40.** In the triangle  $ABC$  (with the usual notation),

$$\text{show that } \frac{r_1 r_2 r_3}{r^3} = \frac{(a+b+c)^3}{(b+c-a)(c+a-b)(a+b-c)}.$$

## CHAPTER XV

### SOLUTION OF TRIANGLES

**95.** In a triangle, there are six parts, the three sides and the three angles. These are not independent, but are connected by the relations between the sides and angles of the triangle, which have been established in Chapter XIV. In fact, if three of the parts are given, the remaining three can, in general, be determined, and the corresponding triangle completely known. The cases that can arise are the following :

- (1) three sides may be given ;
- (2) three angles may be given ;
- (3) two sides and the included angle may be given ;
- (4) two angles and one side may be given ;
- (5) two sides and an opposite angle may be given.

We shall discuss these cases one by one.

#### **96. Three sides given.**

Let the three sides  $a, b, c$  of a triangle  $ABC$  be given. Now, provided the sum of any two of these given sides is greater than the third, the triangle  $ABC$  with the three given sides can be geometrically constructed and the triangle is unique ; in other words, its angles are definite. To determine any angle, say  $A$ , we may use the rigorous formula,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc},$$

and thereby determine  $\cos A$ , and then from the cosine-table find out the angle with this cosine. It is clear that the angle, being an angle of a triangle, lies between 0 and  $\pi$ , and within this range an angle with a given cosine has got only one value. Thus the angle is definitely known.

Here we want to make one point clear. Though the formula used is rigorous, the cosine-table, by means of which we determine the angle with a given cosine, gives only approximate values. Now, it is a principle proved in books on higher mathematics (with the aid of Calculus), that *when an angle is determined by using an approximate table the best result is obtained by using the Logarithmic tangent-table*, and an angle determined from its  $L \tan$ , using a four-figure table is more accurate than that determined by using even a seven-figure sine-table or cosine-table. If a suitable tangent formula is available, therefore, we should make use of it.

Hence, for practical purposes, in this case, to determine  $A$ , we use the formula,

$$\tan \frac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}},$$

where  $s = \frac{1}{2}(a+b+c)$ , which is known.

Taking logarithm, and adding 10, we get the value of  $L \tan \frac{1}{2}A$  and therefore  $A$  is known.

Similarly,  $B$  and  $C$  are determined.

In case  $\tan \frac{1}{2}A$  happens to be equal to the tangent of a standard angle,  $\frac{1}{2}A$  is at once known and the use of logarithm is not wanted.

**Ex.** The sides of a triangle are 2, 3, 4. Find the greatest angle, having given

$$\log 2 = \cdot 30103, \quad \log 3 = \cdot 4771213,$$

$$L \tan 52^\circ 14' = 10^\circ 1108395, \quad L \tan 52^\circ 15' = 10^\circ 1111004.$$

$$\text{Here, } s = \frac{2+3+4}{2} = \frac{9}{2}.$$

The greatest side 4 being denoted by 'a', the greatest angle  $A$  (which is opposite to the greatest side) is given by

$$\tan \frac{1}{2}A = \sqrt{\left(\frac{s}{2} - 2\right)\left(\frac{s}{2} - 3\right)} = \sqrt{\frac{5 \cdot 3}{9 \cdot 1}} = \sqrt{\frac{10}{2 \cdot 3}}.$$

$$\begin{aligned} \therefore L \tan \frac{1}{2}A &= 10 + \frac{1}{2}(\log 10 - \log 2 - \log 3) \\ &= 10 + \frac{1}{2}(1 - \cdot 30103 - \cdot 4771213) \\ &= 10^\circ 1109244. \end{aligned}$$

Now,  $L \tan \frac{1}{2}A$  lies between  $L \tan 52^\circ 14'$  and  $L \tan 52^\circ 15'$ .

Hence,  $\frac{1}{2}A$  lies between  $52^\circ 14'$  and  $52^\circ 15'$ .

Let  $\frac{1}{2}A = 52^\circ 14' x''$ .

Then diff. for  $x''$  is  $\cdot 0000849$ ,

and diff. for  $1'$  i.e.,  $60''$  is  $\cdot 0002609$ .

Hence,  $\frac{x}{60} = \frac{849}{2609}$ , or,  $x = \frac{60 \times 849}{2609} = 19 \cdot 5$  nearly.

Hence,  $\frac{1}{2}A = 52^\circ 14' 19'' \cdot 5$ ,

or,  $A = 104^\circ 28' 39''$  nearly.

### 97. Three angles given.

In this case the triangle cannot be solved, for there are innumerable triangles with the same three angles. All these

triangles, being equiangular, are similar, and the ratio of their sides can be determined from the formula,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C},$$

$$\text{or, } a : b : c = \sin A : \sin B : \sin C.$$

**Ex.** *The angles of a triangle are in the ratio 2 : 3 : 7. Prove that the sides are in the ratio  $\sqrt{2} : 2 : (\sqrt{3} + 1)$ .*

The angles being in the ratio 2 : 3 : 7, and their sum being  $180^\circ$ , the angles are evidently  $30^\circ$ ,  $45^\circ$  and  $105^\circ$  respectively. Hence, the ratio of the sides will be

$$\sin 30^\circ : \sin 45^\circ : \sin 105^\circ,$$

$$\text{i.e., } \frac{1}{2} : \frac{1}{\sqrt{2}} : \frac{\sqrt{3} + 1}{2\sqrt{2}},$$

$$\text{or, } \sqrt{2} : 2 : (\sqrt{3} + 1).$$

### Examples XV(a)

1. The sides of a triangle are 24, 22, 14 ; find the least angle, given  $L \tan 17^\circ 33' = 9.500042$ , diff. for  $1' = 439$ .

2. The sides of a triangle are 50, 36 and 28 ; find the greatest angle, having given

$$\log 10 = 1.2787536, \quad \log 29 = 1.4623980$$

$$L \tan 51^\circ 0' = 10.0916308, \quad L \tan 50^\circ 1' = 10.0918891.$$

3. The sides of a triangle are 9, 10 and 11 ; find the angle opposite to the side 10, given

$$L \tan 29^\circ 30' = 9.7526420, \quad L \tan 29^\circ 29' = 9.7523472,$$

$$\log 2 = .30103. \quad [C. U. 1943]$$

4. The sides of a triangle are 2, 3, 4. Find all the angles correct to degrees and minutes by the help of mathematical tables.

5. (i) The sides of a triangle are 15, 19, 24 ; find the greatest angle of the triangle.

$$\text{Given } \log 5.7 = .75587, L \cos 88^\circ 59' = 8.24903$$

$$\text{diff. for } 1' = 718.$$

[ C. U. 1936 ]

(ii) Find the greatest angle in degrees, minutes and seconds in a triangle whose sides are 5, 6, 7, having given

$$\log 6 = .7781513$$

$$L \cos 39^\circ 14' = 9.8890644, \text{ diff. for } 60'' = .0001032.$$

6. (i) The sides of a triangle are 7, 8, 9 ; solve the triangle.

[ C. U. 1938 ]

(ii) If  $a = 35$ ,  $b = 40$ ,  $c = 66$ , determine the greatest angle.

[ C. U. 1945 ]

[ Use Mathematical Tables ]

7. Given  $a = \sqrt{6}$ ,  $b = 2$ ,  $c = \sqrt{3} - 1$  ; solve the triangle.

8. Given  $a = 2$ ,  $b = \sqrt{2}$ ,  $c = \sqrt{3} + 1$  ; solve the triangle.

9. If  $a = 7$ ,  $b = 5$ ,  $c = 8$ , solve the triangle.

$$\text{Given } \cos 38^\circ 11' = \frac{1}{4}.$$

10. If  $a = 3 + \sqrt{3}$ ,  $b = 2\sqrt{3}$ ,  $c = \sqrt{3}$ , solve the triangle.

11. The angles of a triangle are  $105^\circ$ ,  $60^\circ$  and  $15^\circ$  ; find the ratio of the sides.

12. If  $A = 45^\circ$ ,  $B = 60^\circ$ , show that  $c : a = \sqrt{3} + 1 : 2$ .

13. The angles of a triangle are as  $1 : 2 : 7$  ; find the ratio of the greatest side to the least side.

14. If  $\cos A = \frac{4}{5}$ ,  $\cos B = \frac{3}{5}$ , find  $a : b : c$ .

15. If the angles adjacent to the base of a triangle are  $22\frac{1}{2}^\circ$  and  $112\frac{1}{2}^\circ$ , show that the altitude is half the base.

16. If the sides of a triangle are 4, 5, 6, show that the greatest angle is double the least.



**98. Two sides and the included angle given.**

Let the two sides  $b$ ,  $c$  and the included angle  $A$  of a triangle  $ABC$  be given. It is easy to construct the triangle geometrically, and there will be only one definite triangle with the given parts. To find the other angles  $B$  and  $C$ , we notice that

$$B + C = 180^\circ - A,$$

$$\text{i.e., } \frac{B+C}{2} = 90^\circ - \frac{A}{2}.$$

Again,

$$\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2}.$$

$$\therefore L \tan \frac{B-C}{2} = 10 + \log \left( \frac{b-c}{b+c} \cot \frac{A}{2} \right)$$

$$= \log \left( \frac{b-c}{b+c} \right) + L \cot \frac{A}{2}.$$

$b$ ,  $c$ , and  $A$  being given, the right-hand side is known and thus,  $L \tan \frac{B-C}{2}$  is known, whence  $\frac{B-C}{2}$  is known.

Now  $\frac{B+C}{2}$  and  $\frac{B-C}{2}$  being both known, by addition and subtraction, we get  $B$  and  $C$  respectively.

The reason of using tangent formula to determine  $\frac{B-C}{2}$  is already explained in Art. 96.

When  $B$  and  $C$  are known, the third side  $a$  is easily obtained from

$$\frac{a}{\sin A} = \frac{b}{\sin B}, \text{ or, } = \frac{c}{\sin C}.$$

**Ex.** In a triangle,  $b=2.25$ ,  $c=1.75$ ,  $A=54^\circ$ , find  $B$  and  $C$ , having given,

$$\log 2 = .301030, \quad L \tan 63^\circ = 10.292834$$

$$L \tan 13^\circ 47' = 9.389724, \quad L \tan 13^\circ 48' = 9.390270.$$

[ C. U. 1931 ]

Here,

$$\frac{B+C}{2} = 90^\circ - \frac{A}{2} = 90^\circ - 27^\circ = 63^\circ. \quad (i)$$

Again,

$$\begin{aligned} \tan \frac{B-C}{2} &= \frac{b-c}{b+c} \cot \frac{A}{2} = \frac{5}{4} \cot 27^\circ \\ &= \frac{1}{8} \tan 63^\circ. \end{aligned}$$

$$\begin{aligned} \therefore L \tan \frac{B-C}{2} &= L \tan 63^\circ - 3 \log 2 \\ &= 10.292834 - .903090 \\ &= 9.389744. \end{aligned}$$

Now,  $L \tan 13^\circ 47' = 9.389724$

and  $L \tan 13^\circ 48' = 9.390270$ .

Hence,  $B-C = 1'$

we get, diff. for  $x'' = .000020$

and diff. for  $1'$  i.e.,  $60'' = .000546$ ,

$$\therefore \frac{x}{60} = \frac{.20}{.546}, \quad \text{or,} \quad x = \frac{20 \times 60}{546} = 2.2 \text{ nearly.}$$

Hence,  $\frac{B-C}{2} = 13^\circ 47' 2''.2$  nearly.

Combining with (i),  $B = 76^\circ 47' 2''.2$  and  $C = 49^\circ 12' 57''.8$ .

### 99. Two angles and a side given.

Let any side  $a$  of a triangle  $ABC$ , and any two of its angles be given. The sum of the three angles being  $180^\circ$ , the third angle is also known. To find the other two sides  $b$  and  $c$ , we use the formula,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

**Ex.** In a triangle  $ABC$ ,  $A = 38^\circ 20'$ ,  $B = 45^\circ$  and  $b = 64$  ft. Find  $c$ , having given  $\log 2 = .30103$ ,  $L \sin 83^\circ 20' = 9.99705$  and  $\log .089896 = \bar{2}.95374$ .

Here,

$$C = 180^\circ - (A + B) = 180^\circ - 83^\circ 20'.$$

Now,

$$\frac{c}{\sin C} = \frac{b}{\sin B},$$

$$\text{or, } \frac{c}{\sin (180^\circ - 83^\circ 20')} = \frac{64}{\sin 45^\circ} = \frac{64}{1/\sqrt{2}} = 64\sqrt{2}.$$

$$\therefore c = 2^{\frac{1}{2}} \sin 83^\circ 20'.$$

$$\begin{aligned}\therefore \log c &= \frac{1}{2} \log 2 + L \sin 83^\circ 20' - 10 \\ &= \frac{1}{2} (.30103) + 9.99705 - 10 = 1.95374.\end{aligned}$$

Thus,  $\log c$  has the same mantissa as  $\log .089896$ , but has its characteristic 1. Hence,  $c = 89.896$  feet.

### Examples XV(b)

1. Two sides of a triangle are 3 and 5 feet and the included angle is  $120^\circ$ ; find the other angles, having given

$$\log 4.8 = .6812412$$

$$L \tan 8^\circ 12' = 9.1586706, \text{ diff. for } 60'' = 8940.$$

[ C. U. 1949 ]

2. If  $b = 1300$ ,  $c = 1400$  and  $A = 60^\circ$ , find  $B$  and  $C$ .

$$\text{Given } \log 3 = .4771213,$$

$$L \tan 3^\circ 40' = 8.8067422, \text{ diff. for } 10'' = 3306.$$

3. If  $a = 21$ ,  $b = 11$ ,  $C = 34^\circ 42' 30''$ , find  $A$  and  $B$ .

$$\text{Given } \log 2 = .30103,$$

$$\text{and } L \tan 72^\circ 38' 45'' = 10.50515.$$

4. If the sides  $a$  and  $b$  are in the ratio 7 : 3 and the included angle  $C$  is  $60^\circ$ , find  $A$  and  $B$ , given

$$\log 2 = .3010300, \quad \log 3 = .4771213$$

$$L \tan 34^\circ 42' = 9.8403776, \text{ diff. for } 1' = 2699.$$

5. Two sides of a plane triangle are 14 and 11 and the included angle is  $60^\circ$ . Find the remaining angles, having given  $L \tan 11^\circ 44' = 9.3174299$ ,  $L \tan 11^\circ 45' = 9.3180640$ .

[ C. U. 1922 ]

6. (i) Two sides of a triangle are 80 and 100 ft. and the included angle is  $60^\circ$ . Find the other angles. [ C. U. 1946 ]

(ii) If  $a = 5$ ,  $b = 3$ ,  $C = 70^\circ 30'$ , find the remaining angles.

(iii) If  $a = 39.9$ ,  $b = 43.2$ ,  $C = 38^\circ 14'$ , solve the triangle.

[ Use Mathematical Tables ]

7. (i) In a plane triangle,  $b = 540$ ,  $c = 420$  and  $A = 52^\circ 6'$ ; find  $B$  and  $C$ , having given

$$L \tan 26^\circ 3' = 9.6891430,$$

$$L \tan 14^\circ 20' = 9.4074189,$$

$$L \tan 14^\circ 21' = 9.4079453. \quad [ C. U. 1934 ]$$

(ii) Given  $a = 70$ ,  $b = 35$ ,  $C = 36^\circ 52' 12''$ ,  $\log 3 = 0.4771213$ ,  $L \cot 18^\circ 26' 6'' = 10.4771213$ . Calculate the other two angles  $A$  and  $B$ . [ C. U. 1935, '37 ]

8. If  $a = 2\sqrt{6}$ ,  $c = 6 - 2\sqrt{3}$ ,  $B = 75^\circ$ , solve the triangle.

9. Two sides of a triangle are  $\sqrt{3} + 1$  and  $\sqrt{3} - 1$  and the included angle is  $60^\circ$ ; solve the triangle.

10. (i) If  $a = 2$ ,  $b = 1 + \sqrt{3}$ ,  $C = 60^\circ$ , solve the triangle.

(ii) If  $a = 2$ ,  $b = 4$ ,  $C = 60^\circ$ , find  $A$  and  $B$ .

11. If  $a = 19$ ,  $B = 52^\circ 28'$  and  $C = 93^\circ 40'$ , find  $b$ , having given  $\log 27038 = 4.4319746$ ;  $\log 19 = 1.2787536$ ;

$$\log 27037 = 4.4319585;$$

$$L \sin 52^\circ 28' = 9.8992727, \quad L \sin 33^\circ 52' = 9.7460595.$$

12. If  $B = 45^\circ$ ,  $C = 10^\circ$  and  $a = 200$  ft., find  $b$ , having given

$$\log 2 = .30103, \quad L \sin 55^\circ = 9.9133645,$$

$$\log 1726.4 = 3.2371414, \quad \log 1726.5 = 3.2371666.$$

[ C. U. 1947 ]

13. If  $A = 41^\circ 13' 22''$ ,  $B = 71^\circ 19' 5''$ , and  $a = 55$ , find  $b$ ,  
given  $\log 55 = 1.7403627$ ,  $\log 79063 = 4.8979775$ ,

$$L \sin 41^\circ 13' 22'' = 9.8188779,$$

$$L \sin 71^\circ 19' 5'' = 9.9764927.$$

14. (i) If  $B = 70^\circ 30'$ ,  $C = 78^\circ 10'$ ,  $a = 102$ , solve the triangle.

(ii) If  $a = 39$ ,  $A = 81^\circ 35'$ ,  $B = 27^\circ 55'$ , solve the triangle.

(iii) If  $A = 37^\circ 15'$ ,  $B = 72^\circ 5'$ ,  $a = 75.2$ , find  $b$  and  $c$ .

[ *Mathematical tables should be used* ]

15. If  $A = 75^\circ$ ,  $B = 30^\circ$ ,  $b = \sqrt{8}$ , solve the triangle.

16. If  $A = 30^\circ$ ,  $B = 45^\circ$ ,  $b = 2$ , solve the triangle.

17. In a triangle in which each base angle is double of the third angle, the base is 2 ; solve the triangle.

18. Given  $a = \sqrt{57}$ ,  $A = 60^\circ$ ,  $\Delta = 2\sqrt{3}$ , find  $b$  and  $c$ .

### 100. Two sides and an opposite angle given.

Let the two sides  $b$  and  $c$  in a triangle  $ABC$ , and the angle  $B$  opposite to the side  $b$  be given.

In this case, we get the angle  $C$  from the formula,

$$\frac{\sin C}{c} = \frac{\sin B}{b}, \text{ or, } \sin C = \frac{c \sin B}{b}.$$

Now, three cases may arise, namely.

(i)  $c \sin B > b$ . In this case  $\sin C$  is greater than 1, and so  $C$  cannot be obtained. In fact in this case no triangle is possible.

(ii)  $c \sin B = b$ . Here,  $\sin C$  becomes 1 and therefore,  $C = 90^\circ$ . Thus,  $A = 90^\circ - B$ . We thus get a right-angled triangle with right angle at  $C$ , and the side  $a$  will be obtained from

$$c^2 = a^2 + b^2, \text{ or, } a = \sqrt{c^2 - b^2}.$$

- (iii)  $c \sin B < b$ . In this case  $\sin C$  is less than 1, and hence,  $C$  can be determined. Now, sines of supplementary angles are known to be equal, and an angle of a triangle may be acute or obtuse. We therefore get two supplementary values of  $C$  having the same value for  $\sin C$ . Three sub-cases now arise :

*Sub-case A.* If of the two given sides,  $b > c$ , then  $B > C$ , and therefore the obtuse value of  $C$  becomes inadmissible, for otherwise  $B$  is also obtuse and two angles  $B$  and  $C$  of the triangle become both obtuse. Thus, the only admissible solution is the acute value of  $C$ . Now,  $B$  and  $C$  being both known,  $A$  is obtained from  $A + B + C = 180^\circ$ . The side  $a$  will be known from

$$\frac{a}{\sin A} = \frac{b}{\sin B}, \text{ or, } \frac{c}{\sin C}.$$

Thus, the triangle is uniquely solved.

*Sub-case B.* If  $b = c$ , then  $B = C$ , and here also the obtuse value of  $C$  is inadmissible ; with the acute value of  $C$  the triangle is uniquely solved exactly as in the above case.

*Sub-case C.* If  $b < c$ , then  $B < C$ , so that  $C$  may be either acute or obtuse. Both the supplementary values of  $C$  being admissible now, there will be two possible triangles with the three given parts  $b, c, B$ . Corresponding to each value of  $C$ , the value of  $A$  is determined from  $A + B + C = 180^\circ$ , and then  $a$  is obtained from the formula,

$$\frac{a}{\sin A} = \frac{b}{\sin B}, \text{ or, } \frac{c}{\sin C}.$$

As there are two solutions of the triangle in this case, each equally admissible, this sub-case in the solution of a triangle in which  $b, c, B$  are given and  $b > c \sin B$  and  $b < c$ , is referred to as the **Ambiguous Case** in the solution of triangles.

We may sum up the results as follows :

When in a triangle,  $b, c, B$  are given,

- (i) if  $b < c \sin B$ , no triangle is possible ;
- (ii) if  $b = c \sin B$ , we get a definite right-angled triangle as solution ;
- (iii) if  $b > c$  and therefore necessarily  $> c \sin B$ , we get one definite solution having  $C$  acute ;
- (iv) if  $b = c$  and therefore necessarily  $> c \sin B$ , we get one definite solution having  $C$  acute ;
- (v) if  $b > c \sin B$  but  $< c$ , there are two solutions, and this case is the *ambiguous case*,

### 101. Geometrical treatment of the Ambiguous Case.

To make the ideas clear, we proceed to construct geometrically the triangle in which two sides and an opposite angle, *viz.*,  $b, c$  and  $B$  are given.

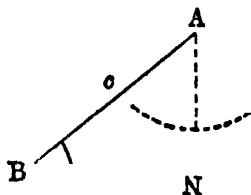


Fig. (i)

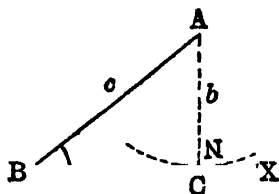


Fig. (ii)

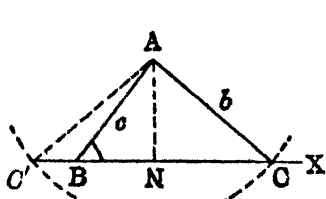


Fig. (iii)

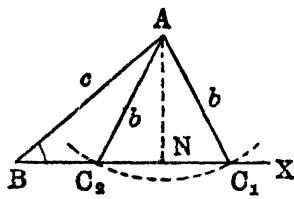


Fig. (iv)

Let  $ABX$  be the given angle  $B$ , and along one arm of it, take  $AB=c$ . Let  $AN$  be the perpendicular from  $A$  on  $BX$ . Then  $\frac{AN}{AB} = \sin B$ , so that  $AN = AB \sin B = c \sin B$ .

With centre  $A$  and radius  $b$  draw a circle.

*Case (i).* If  $b < c \sin B$ , i.e.,  $< AN$ , the circle does not meet the side  $BX$  at all and no triangle is therefore obtained. [ See fig. (i) ]

*Case (ii).* If  $b = c \sin B$ , i.e.,  $= AN$ , the circle touches the side  $BX$  at  $C$  coincident with  $N$ , as in fig. (ii). Hence, a right-angled triangle is formed, in which the sides  $AB$ ,  $AC$  and the angle  $B$  have the given values  $c$ ,  $b$ ,  $B$ . Thus,  $ABC$  is the required triangle.

*Case (iii).* If  $b > c$ , i.e.,  $> AB$ , the circle cuts  $BX$  at two points  $C$  and  $C'$  on opposite sides of  $B$  as in fig. (iii). The triangle  $ABC'$ , though it has the sides  $AB$ ,  $AC$  equal to the given quantities  $c$  and  $b$ , has the angle  $B$  not equal to the given angle. but equal to its supplement. Hence it is not the solution required. In this case the triangle  $ABC$  is the only solution.

*Case (iv).* If  $b = c$ , i.e.,  $= AB$ , the point  $C'$  of the above case coincides with  $B$ , and only one triangle  $ABC$  is obtained as the required solution.

*Case (v).* If  $b > c \sin B$ , i.e.,  $> AN$  but less than  $c$  (or,  $AB$ ), the circle cuts  $BX$  at two points  $C_1$  and  $C_2$  on the same side of  $B$  as in fig. (iv). Both the triangles  $ABC_1$  and  $ABC_2$  have the same three given parts and both are possible solutions. This is therefore the *Ambiguous case*.

**Note.** By considering the equation

$$b^2 = c^2 + a^2 - 2ac \cos B$$



in which  $b, c, B$  are given, we may first of all determine  $a$ , instead of trying to determine  $C$ .

Considering the equation as a quadratic in  $a$ , viz.,

$$a^2 - 2c \cos B \cdot a + c^2 - b^2 = 0,$$

and by solving it, we get

$$a = c \cos B \pm \sqrt{b^2 - c^2 \sin^2 B}.$$

(i) If  $b < c \sin B$ ,  $b^2 - c^2 \sin^2 B$  is negative and thus the two values of  $a$  are imaginary. (No solution)

(ii) If  $b = c \sin B$ ,  $b^2 - c^2 \sin^2 B = 0$  and thus the two values of  $a$  are real and coincident.

(one solution : one triangle right-angled at  $C$ , since  $b = c \sin B$ )

(iii) If  $b > c \sin B$ ,  $b^2 - c^2 \sin^2 B$  is positive, so two values of  $a$  are real and distinct, but they are not always admissible.

(a) When  $b > c$ , i.e.,  $b^2 > c^2 (\sin^2 B + \cos^2 B)$ ,  $b^2 - c^2 \sin^2 B > c^2 \cos^2 B$ , i.e.,  $\sqrt{b^2 - c^2 \sin^2 B} > c \cos B$  and hence one value of  $a$  is positive and the other negative. (one solution)

(b) When  $b = c$ ,  $b^2 - c^2 \sin^2 B = c^2 - c^2 \sin^2 B = c^2 \cos^2 B$  and hence one value of  $a$  is zero. (one solution)

(c) When  $b < c$ , i.e.,  $b^2 < c^2 (\sin^2 B + \cos^2 B)$ ,  $b^2 - c^2 \sin^2 B < c^2 \cos^2 B$ , i.e.,  $\sqrt{b^2 - c^2 \sin^2 B} < c \cos B$ .

So both values of  $a$  are real and positive. (two solutions)

This is known as the *algebraical discussion* of the ambiguous case.

An example illustrating the algebraic method is added below.

**Ex. 1.** In a triangle,  $b = 15$  ft.,  $c = 10$  ft.,  $B = 60^\circ$ . Find  $a$  and  $A$  having given  $\sin 84^\circ 44' = .99578$ .

We have  $b^2 = c^2 + a^2 - 2ca \cos B$ , giving here

$$225 = 100 + a^2 - 20a \cos 60^\circ;$$

or,  $a^2 - 10a - 125 = 0$  whence

$$a = 5 \pm 5\sqrt{6}.$$

Rejecting the negative value for  $a$  as inadmissible, the only possible value of  $a = 5(\sqrt{6} + 1)$  ft. There is thus one solution and there is no ambiguity. In fact this is case (iii) of the previous article.

$$\begin{aligned}\text{Again, } \sin A &= \frac{a}{b} \sin B = \frac{5(\sqrt{6} + 1)}{15} \cdot \frac{\sqrt{3}}{2} = 3 \frac{\sqrt{2} + \sqrt{3}}{6} \\ &= 3 \times 1.41421 \dots + \frac{1.73205}{6} = .99578 \dots\end{aligned}$$

so  $A = 84^\circ 44'$ .

**Ex. 2.** In a triangle,  $a = 73.4$ ,  $b = 64.9$  and  $B = 48^\circ 13' 25''$ ; find  $A$ , having given

$$\log 734 = 2.8656961, \quad \log 649 = 2.8122447$$

$$L \sin 48^\circ 13' 25'' = 9.8725936$$

$$L \sin 57^\circ 30' = 9.9260292 \quad (\text{diff. for } 1' = 804)$$

*Is the case ambiguous?*

Here,

$$\sin A = \frac{a \sin B}{b} = \frac{734}{649} \sin 48^\circ 13' 25''.$$

$$\begin{aligned}\therefore L \sin A &= \log 734 - \log 649 + L \sin 48^\circ 13' 25'' \\ &= 2.8656961 - 2.8122447 + 9.8725936 \\ &= 9.9260450.\end{aligned}$$

Now, diff. of this from  $L \sin 57^\circ 30' = 158$  (i.e., .0000158)  
and diff. for  $1'$  (or  $60''$ ) = 804 (i.e., .0000804).

Hence,  $A = 57^\circ 30' x''$  where

$$\frac{x}{60} = \frac{158}{804} \text{ whence } x = 11.8 \text{ nearly.}$$

Thus,  $A = 57^\circ 30' 11.8''$  or its supplement  $122^\circ 29' 48.2''$   
which has also the same sine, and so the same  $L$  sine.

Now, in this case  $a > b$  and so  $A > B$  and thus both values of  $A$  are admissible. The case is, therefore, the *ambiguous case* and will have two solutions.

**Examples XV(c)**

1. Given (i)  $A = 30^\circ$ ,  $a = 6$ ,  $b = 4$ .  
 (ii)  $A = 60^\circ$ ,  $a = 7$ ,  $b = 8$ .  
 (iii)  $A = 45^\circ$ ,  $a = 2$ ,  $b = 8$ .  
 (iv)  $A = 30^\circ$ ,  $a = 3$ ,  $b = 6$ .

Find in which case the solution is ambiguous, in which case there is one solution, and in which case there is no solution.

2. (i) If  $b = 2$ ,  $c = \sqrt{3} + 1$  and  $B = 45^\circ$ , solve the triangle.

(ii) If  $a = 3$ ,  $b = 3\sqrt{3}$ ,  $A = 30^\circ$ , find  $B$ .

3. If  $a = 2$ ,  $b = \sqrt{6}$ ,  $B = 60^\circ$ , solve the triangle.

4. If  $a = 2$ ,  $b = 5$ ,  $A = 30^\circ$ , solve the triangle.

5. If  $b$ ,  $c$ ,  $B$  are given and if  $b < c$ , show that

$$(a_1 - a_2)^2 + (a_1 + a_2)^2 \tan^2 B = 4b^2,$$

$a_1$  and  $a_2$  being the two possible values of  $a$ .

6. In the ambiguous case, given  $a$ ,  $b$  and  $A$ , prove that the difference between the two values of  $c$  is

$$2\sqrt{a^2 - b^2 \sin^2 A}.$$

7. If  $a$ ,  $b$ ,  $A$  are given, and if  $c_1$ ,  $c_2$  are the values of the third side in the ambiguous case, prove that if  $c_1 > c_2$ ,

$$(i) \quad c_1 - c_2 = 2a \cos B_1. \quad [B. II. U. I. 1928]$$

$$(ii) \quad c_1^2 + c_2^2 - 2c_1 c_2 \cos 2A = 4a^2 \cos^2 A.$$

$$[B. H. U. I. 1935; Pat. I. 1936]$$

$$(iii) \quad \cos \frac{C_1 - C_2}{2} = \frac{b \sin A}{a}. \quad [A. I. 1941]$$

8. If  $b = 16$ ,  $c = 25$  and  $B = 33^\circ 15'$ , find the other angles ; given

$$L \sin 33^\circ 15' = 9.7390129, \quad \log 2 = .30103,$$

$$L \sin 58^\circ 57' = 9.9328376, \quad L \sin 58^\circ 56' = 9.9327616.$$

9. If  $a=5$ ,  $b=4$ ,  $A=45^\circ$ , find  $B$  and  $C$ ; given

$$\log 2 = \cdot 30103, L \sin 34^\circ 27' = 9\cdot 75257.$$

10. If  $a=30$ ,  $b=300$ , find  $A$  in order that  $B$  may be a right angle, having given that

$$L \sin 5^\circ 44' = 8\cdot 9995595, \text{ diff. for } 1' = 12565.$$

11. If  $a=16$ ,  $c=25$  and  $C=60^\circ$ , find the other angles; given

$$\log 2 = \cdot 30103, \quad \log 3 = \cdot 4771213$$

$$L \sin 33^\circ 39' = 9\cdot 7436024, \text{ diff. for } 1' = 1897.$$

12. If  $b=165$ ,  $c=258$ , and  $B=35^\circ 10'$ , find the angles  $A$  and  $C$ ; given

$$\log 1\cdot 65 = \cdot 21749, \quad \log 2\cdot 58 = \cdot 41162$$

$$L \sin 35^\circ 10' = 9\cdot 76039, \quad L \sin 64^\circ 14' = 9\cdot 95452.$$

13. If  $2b=3a$  and  $\tan^2 A = \frac{3}{8}$ , prove that there are two values of the third side, one of which is double the other.

14. If  $A_1$ ,  $B_1$  and  $A_2$ ,  $B_2$  are the angles of the two triangles in the ambiguous case where  $b$ ,  $c$ ,  $C$  are given,

$$\text{then } \frac{\sin A_1}{\sin B_1} + \frac{\sin A_2}{\sin B_2} = 2 \cos C.$$

15. Show that in the case that admits of two solutions, the two values of  $C$  satisfy the equation

$$\frac{(a+b)^2}{1+\cos C} + \frac{(b-a)^2}{1-\cos C} = \frac{2a^2}{\sin^2 A}. \quad [B. H. U. I. 1942]$$

16. If  $\log b + 10 = \log c + L \sin B$ , can the triangle be ambiguous?
-

## Miscellaneous Examples II

In any triangle  $ABC$ , prove that (Ex. 1 to 8):—

1.  $\frac{1}{a} \cos A + \frac{1}{b} \cos B + \frac{1}{c} \cos C = \frac{a^2 + b^2 + c^2}{2abc}$ .
2.  $(b^2 + c^2 - a^2) \tan A = (c^2 + a^2 - b^2) \tan B$   
 $= (a^2 + b^2 - c^2) \tan C$ .
3.  $b^2 + c^2 - 2bc \cos (A + 60^\circ) = c^2 + a^2 - 2ca \cos (B + 60^\circ)$   
 $= a^2 + b^2 - 2ab \cos (C + 60^\circ)$ .
4.  $(\cot \frac{1}{2}A - \tan \frac{1}{2}B - \tan \frac{1}{2}C)^{\frac{1}{2}}$   
 $+ (\cot \frac{1}{2}B - \tan \frac{1}{2}C - \tan \frac{1}{2}A)^{\frac{1}{2}} + (\cot \frac{1}{2}C - \tan \frac{1}{2}A - \tan \frac{1}{2}B)^{\frac{1}{2}}$   
 $= (\cot \frac{1}{2}A + \cot \frac{1}{2}B + \cot \frac{1}{2}C)^{\frac{1}{2}}$ .
5.  $a \sin (B - C) \cos (B + C - A) + b \sin (C - A)$   
 $\times \cos (C + A - B) + c \sin (A - B) \cos (A + B - C) = 0$ .
6.  $\frac{a \sin A + b \sin B + c \sin C}{a \cos A + b \cos B + c \cos C} = \frac{R}{abc} (a^2 + b^2 + c^2)$ .
7.  $(b + c - 2a) \sin \frac{1}{2}A \sin \frac{1}{2}(B - C)$   
 $+ (c + a - 2b) \sin \frac{1}{2}B \sin \frac{1}{2}(C - A)$   
 $+ (a + b - 2c) \sin \frac{1}{2}C \sin \frac{1}{2}(A - B) = 0$ .
8.  $a \cos A \cos 2A + b \cos B \cos 2B + c \cos C \cos 2C$   
 $+ 4 \cos A \cos B \cos C (a \cos A + b \cos B + c \cos C) = 0$ .
9. If in a triangle,  $a^2, b^2, c^2$  are in A.P., show that  
 $\tan A, \tan B, \tan C$  are in H.P.
10. If in a triangle,  $\sin A, \sin B, \sin C$  are in H.P., show  
that  $1 - \cos A, 1 - \cos B, 1 - \cos C$  are in H.P.
11. Determine the triangle whose sides are three consecutive terms in the series of natural numbers and whose largest angle is double the least.

12. If in a triangle,  $\cos 3A + \cos 3B + \cos 3C = 1$ , show that one angle must be  $120^\circ$ .

13. If in a triangle,  $\sin A, \sin B, \sin C$  be in A.P., show that  $\tan \frac{1}{2}A \tan \frac{1}{2}C = \frac{1}{3}$ .

14. If  $a=5$ ,  $b=7$  and  $A=30^\circ$ , find  $B$  in degrees and minutes, having given

$$\sin 44^\circ = 0.6947, \sin 45^\circ = 0.7071.$$

15. In the *ambiguous case*, the area of one of the triangles is  $n$  times that of the other; show that if  $b$  be the greater of the given sides and  $c$  the less,  $\frac{b}{c}$  is less than  $\frac{n+1}{n-1}$ .

16. In the *ambiguous case*, show that the circum-circles of the two triangles are equal.

17. Prove that

$$(i) \tan^{-1} \left( \frac{x \cos \phi}{1 - x \sin \phi} \right) - \tan^{-1} \left( \frac{x - \sin \phi}{\cos \phi} \right) = \phi.$$

$$(ii) \tan^{-1} \frac{t_1 - t_2}{1 + t_1 t_2} + \tan^{-1} \frac{t_2 - t_3}{1 + t_2 t_3} \\ + \dots + \tan^{-1} \frac{t_{n-1} - t_n}{1 + t_{n-1} t_n} = \tan^{-1} t_1 - \tan^{-1} t_n.$$

18. If the sum of four angles be  $180^\circ$ , prove that the sum of the products of their cosines taken two and two together is equal to the sum of the products of their sines taken similarly.

$$19. \text{ Prove that } \cos^2 A + \cos^2 \left( A + \frac{\pi}{3} \right) + \cos^2 \left( A - \frac{\pi}{3} \right) = \frac{3}{2}.$$

[ C. U. 1933 ]

20. In a triangle  $ABC$ , if  $\tan \frac{A}{2}, \tan \frac{B}{2}, \tan \frac{C}{2}$  be in Arithmetical Progression then  $\cos A, \cos B, \cos C$  are also in Arithmetical Progression.

21. Give in general terms the solutions of the following equation :

$$\tan (x+b) \tan (x+c)+\tan (x+c) \tan (x+a) \\ +\tan (x+a) \tan (x+b)=1 .$$

22. If  $A+B+C=180^\circ$ , prove that

$$\left(1+\tan \frac{A}{4}\right)\left(1+\tan \frac{B}{4}\right)\left(1+\tan \frac{C}{4}\right) \\ =2\left(1+\tan \frac{A}{4} \tan \frac{B}{4} \tan \frac{C}{4}\right) .$$

23. Prove that

$$\sin ^2 x+\sin ^2 y+\sin ^2 z+\sin ^2 (x+y+z) \\ =2-2 \cos (x+y) \cos (y+z) \cos (z+x) .$$

24. Solve the following equation :

$$\tan x+\tan \left(x+\frac{\pi}{3}\right)+\tan \left(x+\frac{2 \pi}{3}\right)=3 .$$

[ *Left side reduces to  $3 \tan 3x$ .* ]

25. Prove that in a triangle  $ABC$ ,

$$\Delta=\frac{(a+b+c)^2}{4\left(\cot \frac{A}{2}+\cot \frac{B}{2}+\cot \frac{C}{2}\right)} .$$

26. Prove that

$$\log \sin 8 x=3 \log 2+\log \sin x+\log \cos x \\ +\log \cos 2 x+\log \cos 4 x .$$

27. Show that in any triangle  $ABC$ ,

$$\log \tan \frac{A}{2}=\frac{1}{2}[\log (s-b)+\log (s-c)-\log s-\log (s-a)] .$$

28. Prove that (i)  $x^{\log y}=y^{\log x}$ .

$$(ii) x^{\log y-\log z} \times y^{\log z-\log x} \times z^{\log x-\log y}=1 .$$

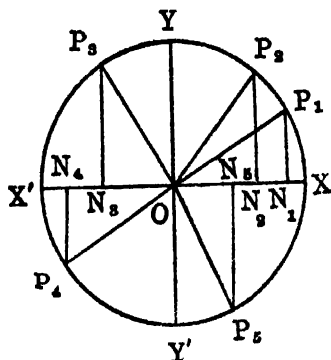
29. In any right-angled triangle  $ABC$ ,  $C$  being the right angle, show that  $R+r=\frac{1}{2}(a+b)$ .

30. Show how to solve a triangle having given the three perpendiculars from the vertices on the opposite sides.

## CHAPTER XVI

### GRAPHS OF TRIGONOMETRICAL FUNCTIONS

**102. Changes in the Trigonometrical ratios of an angle as the angle increases from  $0^\circ$  to  $360^\circ$ .**



Suppose an angle traced out by a revolving line starting from  $OX$ , changes gradually from  $0^\circ$  to  $360^\circ$ .

Take a circle with centre  $O$  of any radius. It is clear that in determining the trigonometrical ratios of an angle  $XOP_1$  in its different positions, we can keep the hypotenuse  $OP_1$  always the same, equal to the radius of the circle.

#### (i) Changes in sine.

When the angle  $N_1OP_1$  ( $=\theta$  say) is zero, its sine is zero. As the angle increases from  $0^\circ$  to  $90^\circ$ , the hypotenuse  $OP_1$  remaining the same, the opposite side  $P_1N_1$  is positive and gradually increases, as is evident by comparing the triangles  $N_1OP_1$  and  $N_2OP_2$ .



Hence,  $\sin \theta = \frac{P_1 N_1}{OP_1}$  gradually increases, until when  $\theta = 90^\circ$ ,  $P_2 N_2$  and  $OP_2$  both coincide with  $OY$  and  $\sin \theta$  attains its greatest value 1.

As  $\theta$  still further increases, from  $90^\circ$  to  $180^\circ$ , the hypotenuse  $OP_3$  retains the same value, but  $P_3 N_3$  remaining positive, now gradually diminishes from  $OY$  to zero, and so  $\sin \theta$  diminishes from 1 to 0. In the third quadrant, as  $\theta$  increases from  $180^\circ$  to  $270^\circ$ ,  $P_4 N_4$  is negative and numerically increases from zero to  $OY'$ , the hypotenuse remaining always positive and unaltered.  $\sin \theta$  is therefore negative and numerically increases from 0 to 1; in other words, it diminishes gradually from 0 to -1. In the fourth quadrant, as  $\theta$  increases from  $270^\circ$  to  $360^\circ$ ,  $P_5 N_5$  remaining negative numerically diminishes from  $OY'$  to 0, and  $\sin \theta$  therefore remaining negative numerically diminishes from 1 to 0; in other words, it increases from -1 to 0. The results are therefore as follows :

In the *first* quadrant, as  $\theta$  increases from  $0^\circ$  to  $90^\circ$ ,

$\sin \theta$  increases from 0 to 1.

In the *second* quadrant, as  $\theta$  increases from  $90^\circ$  to  $180^\circ$ ,

$\sin \theta$  diminishes from 1 to 0.

In the *third* quadrant, as  $\theta$  increases from  $180^\circ$  to  $270^\circ$ ,

$\sin \theta$  diminishes from 0 to -1.

In the *fourth* quadrant, as  $\theta$  increases from  $270^\circ$  to  $360^\circ$ ,

$\sin \theta$  increases from -1 to 0.

## (ii) Changes in cosine.

In the first quadrant, as the angle  $XOP_1$  increases,  $ON_1$  diminishes, from the value of  $OX$  at  $\theta = 0^\circ$  to the value 0 at  $\theta = 90^\circ$ , and is always positive.

In the second quadrant, as  $\theta$  goes on increasing from  $90^\circ$  to  $180^\circ$ ,  $ON_2$  increases numerically from 0 to  $OX'$  but is

negative. In the third quadrant,  $ON_4$  remains negative, but diminishes numerically from  $OX'$  to 0. In the fourth quadrant,  $ON_5$  is positive and increases from 0 to  $OX$  again.

The hypotenuse remains always positive and is equal to  $OX$  or  $OX'$  in magnitude.

We thus come to the conclusions :

As  $\theta$  increases from  $0^\circ$  to  $90^\circ$ ,

$\cos \theta$  diminishes from 1 to 0.

As  $\theta$  increases from  $90^\circ$  to  $180^\circ$ ,

$\cos \theta$  diminishes from 0 to  $-1$ .

As  $\theta$  increases from  $180^\circ$  to  $270^\circ$ ,

$\cos \theta$  increases from  $-1$  to 0.

As  $\theta$  increases from  $270^\circ$  to  $360^\circ$ ,

$\cos \theta$  increases from 0 to 1.

### (iii) Changes in tangent.

As  $\theta$  goes on increasing from  $0^\circ$  to  $90^\circ$  in the first quadrant,  $P_1N_1$  increases from 0 to  $OY$  and simultaneously  $ON_1$  decreases from  $OX$  to 0, both remaining positive ; hence,

$\tan \theta = \frac{P_1N_1}{ON_1}$  increases from the value  $\frac{0}{OX} = 0$  to  $\frac{OY}{0} \rightarrow \infty$ .

In the second quadrant,  $P_2N_2$  diminishes from  $OY$  to 0 while  $ON_2$ , becoming negative, numerically increases from 0 to  $OX'$ . Hence,  $\tan \theta = \frac{P_2N_2}{ON_2}$  is negative but numerically diminishes from  $\infty$  to 0, i.e., increases from  $-\infty$  to 0.

Immediately before  $90^\circ$ ,  $\tan \theta$  is positive and very large, while immediately after  $90^\circ$   $\tan \theta$  is negative and numerically very large. In fact, here, as  $\theta$  passes through the value  $90^\circ$  from the first to the second quadrant, there is

a sudden break or discontinuity in the value of  $\tan \theta$ , which suddenly passes from a very large positive value to a very large negative value, *i.e.*, from the positive to the negative side in passing through infinity.

In the third quadrant, both  $P_4N_4$  and  $ON_4$  are negative and  $P_4N_4$  increases numerically from 0 to  $OY'$  while  $ON_4$  decreases numerically from  $OX'$  to 0. Hence,  $\tan \theta = \frac{P_4N_4}{ON_4}$  is positive and increases from 0 to  $\infty$ .

In the fourth quadrant,  $P_5N_5$  is negative and numerically diminishes from  $OY'$  to 0 while  $ON_5$  is positive and increases from 0 to  $OX$ . Hence,  $\tan \theta = \frac{P_5N_5}{ON_5}$  is negative and numerically diminishes from  $\infty$  to 0, *i.e.*, increases from  $-\infty$  to 0.

In passing through  $270^\circ$ , there is another discontinuity,  $\tan \theta$  suddenly passing from the positive to the negative side through infinity.

The results are therefore as follows :

As  $\theta$  increases from  $0^\circ$  to  $90^\circ$ ,  $\tan \theta$  increases from 0 to  $\infty$

As  $\theta$  passes through  $90^\circ$ ,  $\tan \theta$  suddenly changes from  
 $+\infty$  to  $-\infty$

As  $\theta$  increases from  $90^\circ$  to  $180^\circ$ ,  $\tan \theta$  increases from  
 $-\infty$  to 0

As  $\theta$  increases from  $180^\circ$  to  $270^\circ$ ,  $\tan \theta$  increases from  
0 to  $\infty$

As  $\theta$  passes through  $270^\circ$ ,  $\tan \theta$  suddenly changes from  
 $+\infty$  to  $-\infty$

As  $\theta$  increases from  $270^\circ$  to  $360^\circ$ ,  $\tan \theta$  increases from  
 $-\infty$  to 0.

**(iv) Changes in cotangent.**

From the changes in the value of the tangent the changes in  $\cot \theta = \frac{1}{\tan \theta}$  are traced as follows :

$\theta$  increasing from  $0^\circ$  to  $90^\circ$ ,  $\cot \theta$  diminishes from  $\infty$  to 0

$\theta$  increasing from  $90^\circ$  to  $180^\circ$ ,  $\cot \theta$  diminishes from 0 to  $-\infty$

As  $\theta$  passes through  $180^\circ$ , there is a sudden change in  $\cot \theta$  from  $-\infty$  to  $+\infty$

$\theta$  increasing from  $180^\circ$  to  $270^\circ$ ,  $\cot \theta$  diminishes from  $+\infty$  to 0

$\theta$  increasing from  $270^\circ$  to  $360^\circ$ ,  $\cot \theta$  diminishes from 0 to  $-\infty$

As  $\theta$  passes through  $360^\circ$ ,  $\cot \theta$  again suddenly changes from  $-\infty$  to  $+\infty$ .

**(v) Changes in secant.**

For  $\sec \theta = \frac{1}{\cos \theta}$ , the results are as follows :

From  $0^\circ$  to  $90^\circ$  for  $\theta$ ,  $\sec \theta$  increases from 1 to  $\infty$ .

Here, there is a sudden change from  $+\infty$  to  $-\infty$ .

Then from  $90^\circ$  to  $180^\circ$ ,  $\sec \theta$  increases from  $-\infty$  to  $-1$ .

From  $180^\circ$  to  $270^\circ$ ,  $\sec \theta$  diminishes from  $-1$  to  $-\infty$ .

Here, again there is a sudden change from  $-\infty$  to  $+\infty$ .

Then from  $270^\circ$  to  $360^\circ$ ,  $\sec \theta$  diminishes from  $\infty$  to 1.

**(vi) Changes in cosecant.**

For  $\operatorname{cosec} \theta = \frac{1}{\sin \theta}$ , the results are as follows :

From  $0^\circ$  to  $90^\circ$ ,  $\operatorname{cosec} \theta$  diminishes from  $\infty$  to 1.

From  $90^\circ$  to  $180^\circ$ ,  $\operatorname{cosec} \theta$  increases from 1 to  $\infty$ .

Here,  $\operatorname{cosec} \theta$  suddenly changes from  $+\infty$  to  $-\infty$ .

Then from  $180^\circ$  to  $270^\circ$ ,  $\operatorname{cosec} \theta$  increases from

$-\infty$  to  $-1$ .

From  $270^\circ$  to  $360^\circ$ ,  $\operatorname{cosec} \theta$  diminishes from  $-1$  to  $-\infty$ .

As  $\theta$  passes through  $360^\circ$ ,  $\operatorname{cosec} \theta$  again suddenly

changes from  $-\infty$  to  $+\infty$ .

**Note.** As  $\theta$  increases by complete multiples of  $2\pi$  (i.e.,  $360^\circ$ ) we know that all the trigonometrical ratios remain unaltered. Hence after  $360^\circ$ , as  $\theta$  goes on increasing, the same series of values for the ratios are repeated over and over again for each complete revolution of the revolving line. The trigonometrical ratios are therefore all of them **periodic functions** having the same period  $2\pi$ ,\* after each of which the same cycle of values is repeated.

The changes traced out above, of the trigonometrical ratios, may be much more clearly demonstrated to the eye from a study of their graphs.

### 103. Graphs of Trigonometrical Functions.

Just like algebraic functions, trigonometrical functions (i.e.,  $\sin x$ ,  $\cos x$ ,  $\sin^2 2x + \tan \frac{x}{2}$ , etc.) may be conveniently represented by means of graphs, showing their changes with the change in the values of the angles.

The method is the same as for graphs in Algebra. Two straight lines  $XOX'$  and  $YOY'$ , intersecting at right angles are taken as axes of co-ordinates. Along the  $x$ -axis, the angles are represented on a suitably chosen scale, positive angles along  $OX$ , and negative angles along  $OX'$ . Along the  $y$ -axis the values of the trigonometrical functions corresponding to the angles are represented on a suitably chosen scale, positive values being measured upwards (along  $OY$ ), and negative values downwards (along  $OY'$ ). Thus, the *abscissa* and *ordinate* of a point stand respectively for an angle and its trigonometrical function.

\*  $\tan \theta$  and  $\cot \theta$  have a period  $\pi$ .

Plotting a number of points in this way and joining them free-hand, we get the required graph of a given trigonometrical function.

#### 104. Graph of $\sin x$ or sine-graph.

Let  $y = \sin x$ .

Using the table of natural sines, the corresponding values of  $x$  and  $y$  are tabulated corresponding to the values of  $x$  differing by  $10^\circ$  (the values of  $y$  being correct up to two places of decimals) as follows :—

	$-90^\circ$	$-80^\circ$	$-70^\circ$	$-60^\circ$	$-50^\circ$	$-40^\circ$	$-30^\circ$	$-20^\circ$	$-10^\circ$	$0^\circ$
$\frac{y}{\sin x}$ or	-1	·98	·94	·87	·77	·64	·50	·34	·17	0

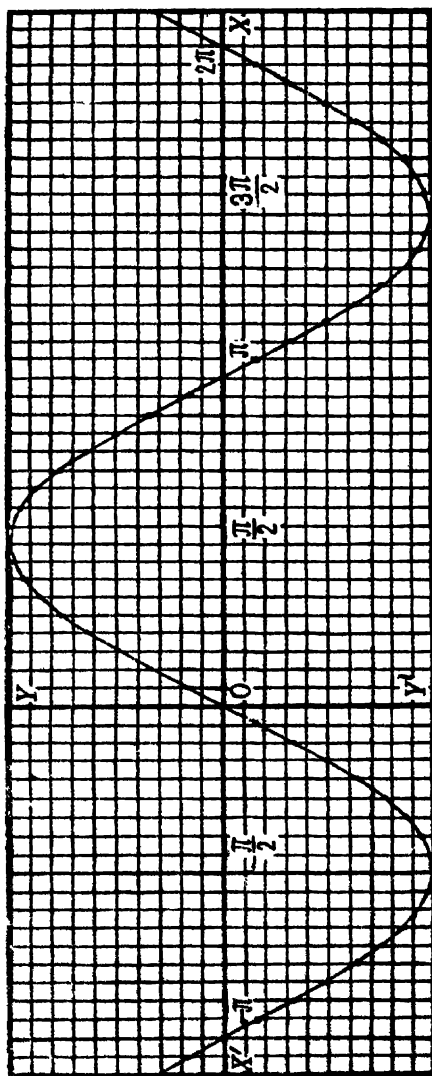
$x$	$10^\circ$	$20^\circ$	$30^\circ$	$40^\circ$	$50^\circ$	$60^\circ$	$70^\circ$	$80^\circ$	$90^\circ$	$100^\circ$	$110^\circ$	$120^\circ$	etc.
$\frac{y}{\sin x}$ or	·17	·34	·50	·64	·77	·87	·94	·98	1	·98	·94	·87	etc.

Now, let the scale be so chosen that 1 small division along  $OX$  represents  $10^\circ$ , and 10 small divisions along  $OY$  represent unity.\*

The points corresponding to the tabulated values are plotted on the graph paper according to the scale chosen and joined free-hand.

The graph is as shown on the next page (drawn here between the range  $x = -180^\circ$  to  $x = +360^\circ$ ).

\*According to the graph paper supplied and the range within which the graph is to be drawn, the scale should be suitably chosen in each individual case separately.



Sine graph

**Note 1.** In the table of natural sines, sines of angles from  $0^\circ$  to  $90^\circ$  only are available. With the help of the formulae  $\sin(-\theta) = -\sin \theta$ ,  $\sin(180^\circ - \theta) = \sin \theta$ ,  $\sin(180^\circ + \theta) = -\sin \theta$  etc. of Chapter IV, however, the tabulation for  $\sin \theta$  shown above, outside the range of  $0^\circ$  to  $90^\circ$  is effected.

Similar is the case of tabulation for other graphs in the following pages.

**Note 2.** *Peculiarities of the sine-graph.*

From the figure, the following features will be apparent :—(i) the graph is continuous, and wavy in form ; (ii) the maximum value of  $\sin x$  is  $+1$  and the minimum value is  $-1$ , these values being attained for values of  $x$  which are odd multiples of  $90^\circ$  ; (iii)  $\sin x$  is 0 at the origin and at points for which  $x$  is an even multiple of  $90^\circ$  i.e., any multiple of  $180^\circ$  ; (iv) that  $\sin\left(\frac{\pi}{2} - x\right) = \sin\left(\frac{\pi}{2} + x\right)$ ,  $\sin(\pi - x) = \sin x$ ,  $\sin(-x) = -\sin x$ ,  $\sin(\pi + x) = -\sin x$  etc. ; (v) since  $\sin(2\pi + x) = \sin x$ , the portion between 0 to  $2\pi$  is repeated over and over again on either side.

### 105. Graph of $\cos x$ or cosine-graph.

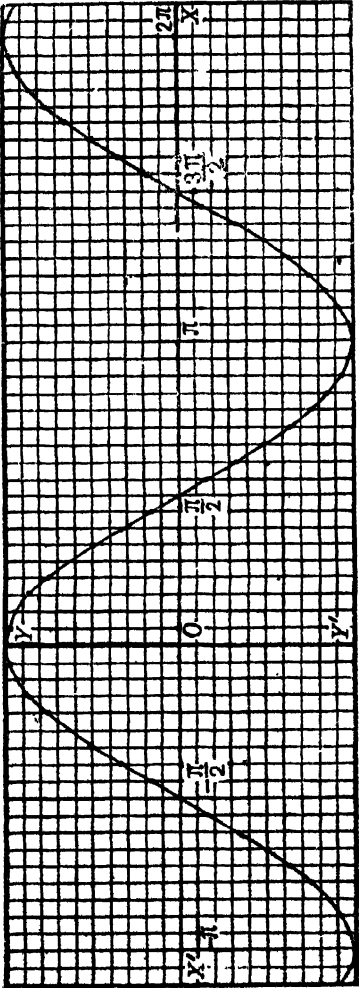
Let  $y = \cos x$ .

Using the table of natural cosines (see *Note 1 of the previous Article*), the corresponding values of  $x$  and  $y$  are tabulated at intervals of  $10^\circ$  for  $x$  as follows :—

$x$	$-90^\circ$	$-80^\circ$	$-70^\circ$	$-60^\circ$	$-50^\circ$	$-40^\circ$	$-30^\circ$	$-20^\circ$	$-10^\circ$
$y$ or $\cos x$	0	.17	.34	.50	.64	.77	.87	.94	.98

$x$	$0^\circ$	$10^\circ$	$20^\circ$	$30^\circ$	$40^\circ$	$50^\circ$	$60^\circ$	$70^\circ$	$80^\circ$	$90^\circ$	$100^\circ$	$110^\circ$	etc.
$y$ or $\cos x$	1	.98	.94	.87	.77	.64	.50	.34	.17	0	-.17	-.34	etc.





Cosine-graph

Now, choosing the scale such that 1 small division along  $OX$  represents  $10^\circ$ , and 10 small divisions along  $OY$  represent unity, the points corresponding to the above tabulated values are plotted and joined free-hand.

We then get the required graph, which is shown on the annexed page (shown here between the range  $-\pi$  to  $+2\pi$  of  $x$ ).

**Note.** It is apparent from the figure, that the *cosine-graph* is exactly the same as the *sine-graph* only shifted wholesale backwards (to the left) through a space of  $90^\circ$ .

This is due to the fact that  $\sin(90^\circ + x) = \cos x$ , or  $\sin x = \cos(x - 90^\circ)$  so that the ordinate in the sine-graph corresponding to any value of  $x$  = the ordinate of the cosine-graph corresponding to a value of  $x$  which is  $90^\circ$  less than before.

### 106. Graph of $\tan x$ or *tangent-graph*.

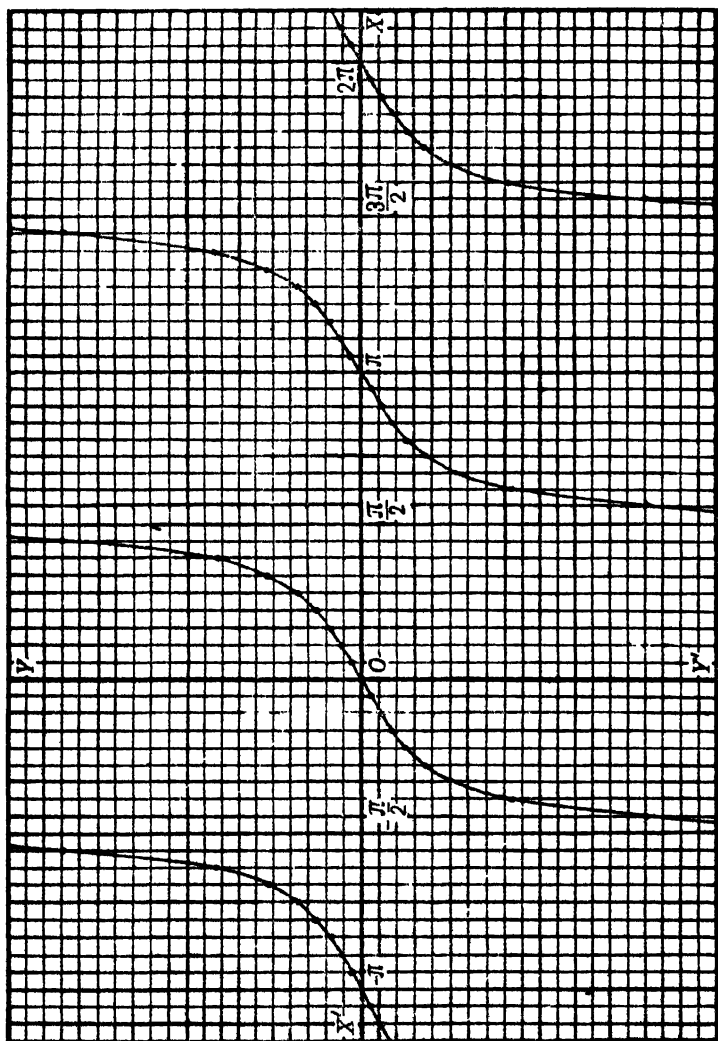
Let  $y = \tan x$ .

Using the table of natural tangents, the corresponding values of  $x$  and  $y$  are tabulated at intervals of  $10^\circ$  of  $x$  as follows :—

$x$	$-20^\circ$	$-10^\circ$	$0^\circ$	$10^\circ$	$20^\circ$	$30^\circ$	$40^\circ$	$50^\circ$	$60^\circ$	$70^\circ$	$80^\circ$	$90^\circ$	$100^\circ$	etc.
$y$ or $\tan x$	$-.36$	$-.18$	$0$	$.18$	$.36$	$.58$	$.84$	$1.19$	$1.73$	$2.75$	$5.67$	$\infty$	$-5.67$	etc.

Now, choosing the scale such that 1 small division along  $OX$  represents  $10^\circ$ , and 3 small divisions along  $OY$  represent unity, the points corresponding to the above tabulated values are plotted and joined free-hand.

The graph is as shown on the next page (shown here between the range  $-\pi$  to  $+2\pi$  for  $x$ ).



Tangent-graph

**Note.** *Peculiarities of the tangent-graph.*

From the figure, the following points will be apparent : (i) That the curve is not continuous, but consists of separate branches or portions, the points of discontinuities being the values of  $x$  corresponding to the odd multiples of  $\frac{\pi}{2}$ . (ii) As  $x$  passes through these points from the left to the right, the value of  $\tan x$  suddenly changes from very large positive values on the left to very large negative values on the right. (iii) The lines parallel to  $y$ -axis corresponding to the odd multiple of  $\frac{\pi}{2}$  are continuously approached by the graph on either side, but never actually meet. Such lines are called *asymptotes* to the curve. (iv) Since  $\tan (n\pi + x) = \tan x$ , each branch is simply a repetition of the branch from  $-\frac{\pi}{2}$  to  $+\frac{\pi}{2}$ .

**107. Graph of  $\cot x$  or *cotangent-graph*.**

As before the values of  $x$  and  $y$  ( $= \cot x$ ) are tabulated, and with the same scale as in the tangent-graph the points are plotted and joined free-hand

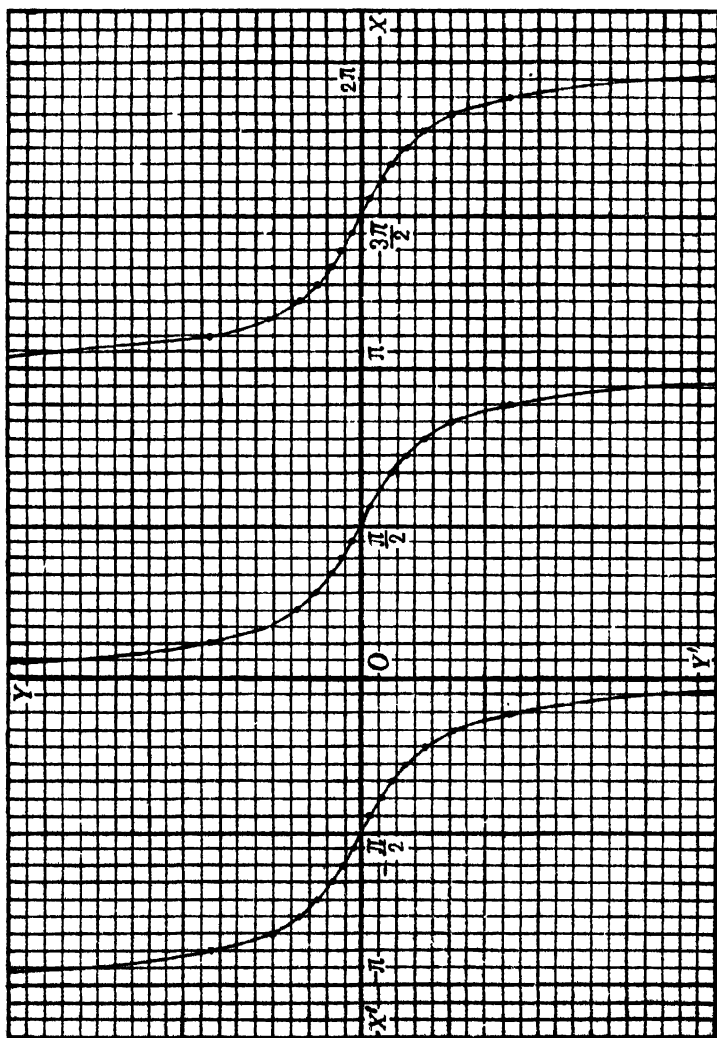
The graph is shown on the next page (between  $x = -\pi$  to  $x = +2\pi$ ).

This graph also, like the tangent-graph, is *discontinuous*, the points of discontinuity being  $x=0$  and  $x=n\pi$ . The portion between  $x=0$  and  $x=\pi$  is repeated over and over again on either side, as is consequent from the formula  $\cot (n\pi + x) = \cot x$ .

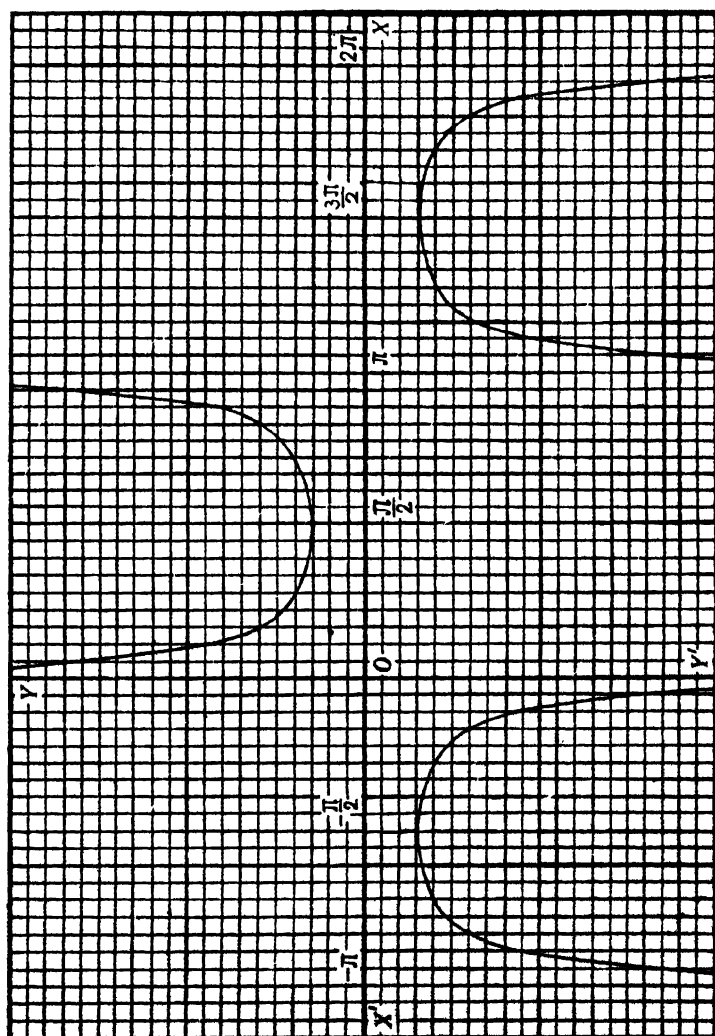
**108. Graph of  $\operatorname{cosec} x$  or *cosecant-graph*.**

The corresponding values of  $x$  and  $y$  are tabulated at intervals of  $10^\circ$  of  $x$  as follows :—

	-20°	-10°	0°	10°	20°	30°	etc.	80°	90°	100°	110°	etc.
$\frac{y}{\operatorname{cosec} x}$ or	-2.92	-5.76	$\infty$	5.76	2.92	2	etc.	1.02	1	1.02	1.06	etc.



Cotangent-graph



Cosecant-graph

[ If the table of natural cosecants be not available, the table of natural sines may be used and the values of  $\operatorname{cosec} x = \frac{1}{\sin x}$  may be calculated for different values of  $x$ . ]

The scale is so chosen that 1 small division along  $OX$  represents  $10^\circ$ , and 3 small divisions along  $OY$  represent unity.

The tabulated points are now plotted and joined free-hand.

The graph is shown on the previous page (between the range  $x = -\pi$  to  $x = 2\pi$ ).

**Note.** This graph also consists of *detached branches*, the points of discontinuity being  $x=0$  and  $x=n\pi$ . The value of  $y$  never lies between  $\pm 1$ , being always greater than 1 or less than  $-1$ . The lines  $x=n\pi$  are asymptotes. The portion between  $x=0$  to  $x=2\pi$  is repeated on either side, over and over again.

### 109. Graph of $\sec x$ or *secant-graph*.

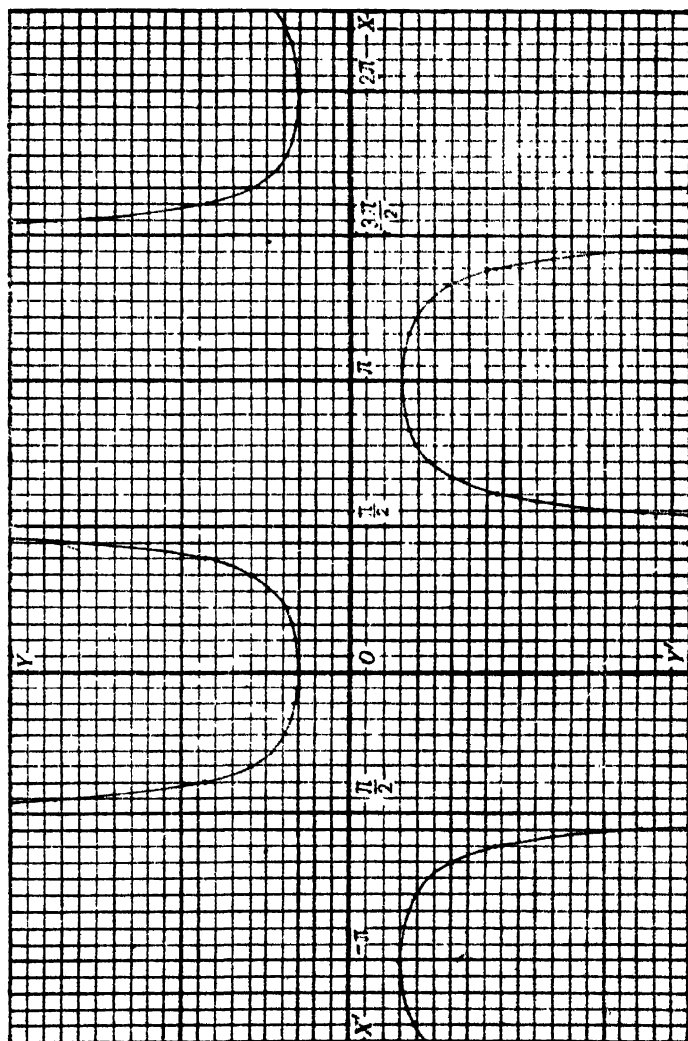
The corresponding values of  $x$  and  $y$  ( $=\sec x$ ) are tabulated as in the case of cosecant-graph, by making use of the table of cosines, if a table of secants be not available.

With the same scale as in the cosecant-graph, the tabulated points have been plotted and joined free-hand.

The graph is shown in the adjoining page (between the range  $x = -\pi$  to  $x = 2\pi$ ).

**Note.** It is apparent from the figure that the *secant-graph* is *exactly the same as the cosecant-graph, only shifted backwards (to the left) through a space of  $90^\circ$* .

This is due to the fact that  $\operatorname{cosec}(90^\circ + x) = \sec x$ . [ See *note below Art. 105* ]



Secant-graph



### 110. Graphs of other Trigonometrical Expressions.

Graphs of other trigonometrical functions may be obtained in a similar manner.

We illustrate this by an example.

**Ex.** Draw the graph of  $y = \sin x + \cos x$  between the range  $x = 0$  to  $x = 2\pi$ , and find from the graph the values of  $x$  for which (i)  $y = 0$ , (ii)  $y$  is maximum, (iii)  $y$  is minimum.

[ C. U. 1934 ]

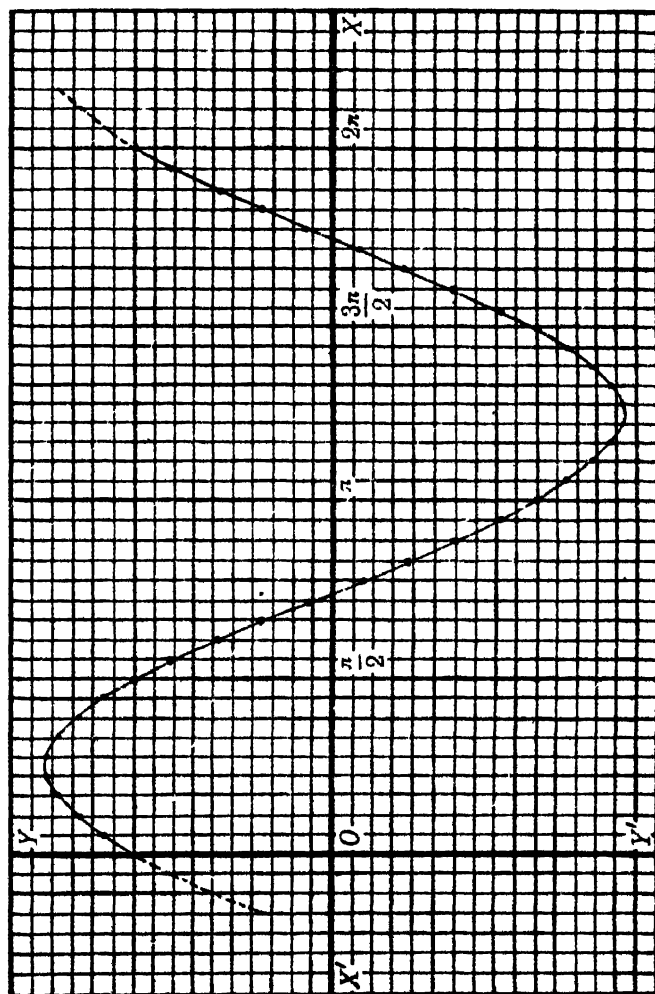
From the table of natural sines and cosines, corresponding to each value of  $x$ , the values of  $\sin x$  and  $\cos x$  may be separately obtained and then added to give  $y$ ; or else we may write  $y = \sin x + \cos x = \sqrt{2} (\sin x \cos \frac{1}{2}\pi + \cos x \sin \frac{1}{2}\pi) = \sqrt{2} \sin(x + \frac{1}{2}\pi)$ , and corresponding to any value of  $x$ ,  $\sin(x + \frac{1}{2}\pi)$  may be deduced from the sine-table, and this multiplied by  $\sqrt{2} (= 1.414)$  will give  $y$ .

The corresponding values of  $x$  and  $y$  are tabulated at intervals of  $10^\circ$  of  $x$ , between the interval  $x = 0$  to  $x = 2\pi$  as follows :—

$x$	$0^\circ$	$10^\circ$	$20^\circ$	$30^\circ$	$40^\circ$	$50^\circ$	$60^\circ$	$70^\circ$	$80^\circ$	$90^\circ$	$100^\circ$
$y$	1	1.15	1.27	1.37	1.41	1.41	1.37	1.27	1.15	1	.81

$x$	$110^\circ$	$120^\circ$	$130^\circ$	$140^\circ$	$150^\circ$	$160^\circ$	$170^\circ$	$180^\circ$	$190^\circ$	$200^\circ$
$y$	.59	.37	.18	-.13	-.37	-.59	-.81	-1	-1.15	-1.27



Graph of  $\sin x + \cos x$

$x$	210°	220°	230°	240°	250°	260°	270°	280°
$y$	-1.37	-1.41	-1.41	-1.37	-1.27	-1.15	-1	-.81

$x$	290°	300°	310°	320°	330°	340°	350°	360°
$y$	-.59	-.37	-.13	.13	.37	.59	.81	1

The scale is chosen so that 1 small division along  $OX$  represents  $10^\circ$ , and 10 small divisions along  $OY$  represent unity.

The tabulated points are now plotted and joined. The graph is as shown on the previous page.

From the graph we find that (i)  $y=0$  when  $x=135^\circ$ , and  $x=315^\circ$ , (ii)  $y$  is maximum when  $x=45^\circ$ , (iii)  $y$  is minimum when  $x=225^\circ$ .

### 111. Graphical solution of Equations.

Trigonometrical equations, just like algebraic equations may be solved graphically. In fact in many practical cases, particularly where the solutions are not obtained in terms of the standard angles, the graphical method of solution is the only one which is found convenient and is actually adopted. The method is illustrated by the following examples.

**Ex. 1.** Solve graphically the equation  $2 \sin^2 x = \cos 2x$ , giving only those solutions of  $x$  which lie between  $-\frac{\pi}{2}$  and  $\frac{3\pi}{2}$ .

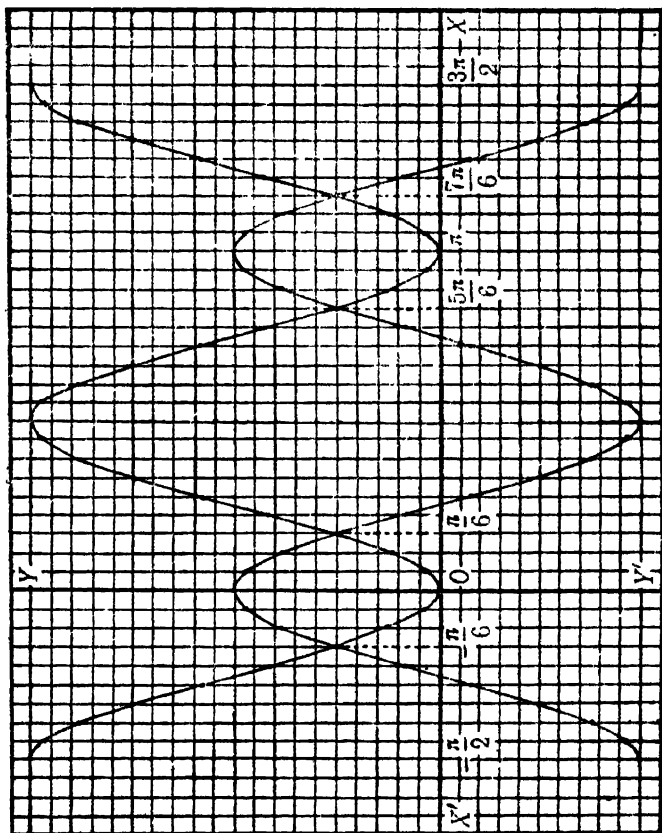
[ C. U. 1938, '46, '48 ]

We draw two graphs, namely

$$y = 2 \sin^2 x = (1 - \cos 2x)$$

$$\text{and } y = \cos 2x$$

by tabulating the corresponding values of  $x$  and  $y$  for the two cases separately, making use of the table of natural



Graphical solution of  $2 \sin^2 x = \cos 2x$ .

cosines, for the range  $x = -\frac{\pi}{2}$  to  $x = \frac{3\pi}{2}$ , at intervals of  $10^\circ$  or  $15^\circ$  of  $x$ .

With the same scale, namely, 1 small division along  $OX$  representing  $10^\circ$ , and 10 small divisions along  $OY$  representing unity, we plot the tabulated values for the two cases in the same graph paper, and joining them, we get the two graphs, as shown in the adjoining page.

We find that the two graphs intersect, and thus have the same abscissæ and ordinates at the points for which

$$x = -\frac{\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}.$$

Thus,  $2 \sin^2 x = \cos 2x$  is satisfied for the values of  $x$  given by

$$x = -\frac{\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6} \text{ and } \frac{7\pi}{6}$$

which are the required solutions within the range

$$-\frac{\pi}{2} \text{ to } \frac{3\pi}{2}.$$

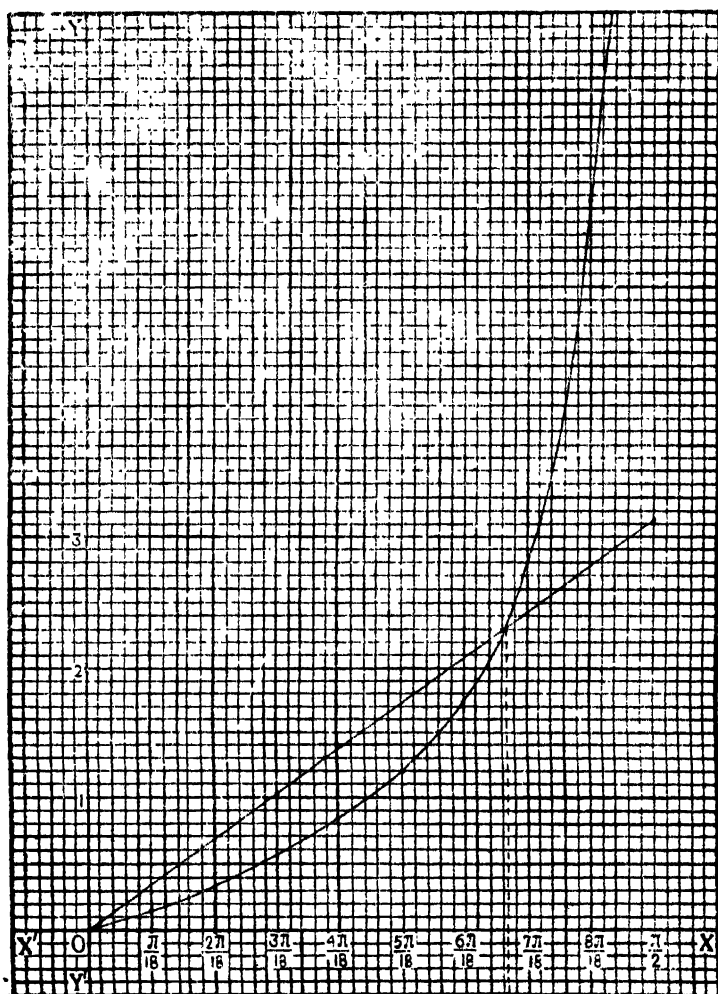
**Ex. 2.** Solve graphically the equation  $\tan x = 2x$  between  $x = 0$  and  $x = \frac{\pi}{2}$ . [ C. U. 1939 ]

Here,  $x$  is supposed to be measured in radians.

First of all we draw separately two graphs, namely

$$y = 2x \quad \dots \quad \dots \quad (i)$$

$$\text{and} \quad y = \tan x \quad \dots \quad \dots \quad (ii)$$



Graphical solution of  $\tan x = 2x$ .

The corresponding values of  $x$  and  $y$  within the range  $x=0$  and  $x=\frac{\pi}{2}$  are tabulated in case (i) as follows :

$x$ (in radians)	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$y$ (i.e. $2x$ ) (numerical value)	0	1.05	2.10	3.15

and in case (ii) as follows :

$x$ (in radians)	0	$\frac{\pi}{18}$	$\frac{2\pi}{18}$	$\frac{3\pi}{18}$	$\frac{4\pi}{18}$	$\frac{5\pi}{18}$	$\frac{6\pi}{18}$	$\frac{7\pi}{18}$	$\frac{8\pi}{18}$	$\frac{\pi}{2}$
$y$ (i.e. $\tan x$ ) (numerical value)	0	.18	.36	.57	.84	1.19	1.73	2.75	5.67	$\infty$

Now, choosing the same scale, namely 5 small divisions along  $OX$  to represent  $\frac{\pi}{18}$  radians, and 10 small divisions along  $OY$  to represent unity, we plot the tabulated points for the two cases in the same graph paper and joining them we get the two graphs within the range  $x=0$  and  $x=\frac{\pi}{2}$  as shown in the adjoining page.

We find that the two graphs intersect at the point given by  $x=0$  and also at the point corresponding to 33.5 small divisions along  $OX$ , which, from our chosen scale, represents  $x = \frac{33.5}{5} \times \frac{\pi}{18}$  radians = 1.17 radians (approximately).

Hence, the given equation  $\tan x = 2x$  is satisfied between  $x=0$  and  $x=\frac{\pi}{2}$  by the values of  $x$ , namely  $x=0$  and  $x=1.17$  (approximately), which are the required solutions in radians.

### Examples XVI

1. Draw the graphs of :

(i)  $\sin 3x$  between  $x=0^\circ$  to  $x=180^\circ$ .

(ii)  $\tan \frac{3}{2}x$  between  $x = -\frac{1}{2}\pi$  to  $x = \pi$ .

(iii)  $\sin \theta \cos \theta$  between  $\theta = -\pi$  to  $\theta = +\pi$ .

(iv)  $\frac{1}{\cos^2 \theta - \sin^2 \theta}$  between  $\theta = -\frac{\pi}{2}$  to  $+\frac{\pi}{2}$ .

(v)  $\cos (\pi \sin x)$  between  $x=0$  to  $x=\frac{1}{2}\pi$ .

(vi)  $\sin \theta - \sqrt{3} \cos \theta$  between  $\theta=0$  to  $\theta=\pi$ .

(vii)  $\frac{1}{2} \operatorname{cosec} \frac{1}{2}x$  between  $x=0$  to  $x=2\pi$ .

2. (i) Trace the changes in the sign of  $\cos \theta - \sin \theta$  as  $\theta$  changes from  $0^\circ$  to  $360^\circ$ . Verify your conclusions by a graph.

(ii) Trace the changes in sign and magnitude of  
 $2 \sin \theta - \sin 2\theta$   
 $2 \sin \theta + \sin 2\theta$  [ *B. H. U. 1931* ]

3. Draw the graph of  $y = \sin (x + \frac{1}{2}\pi)$  between the limits  $x = -\pi$  and  $x = +\pi$ .

4. Draw the graphs of  $\sin \theta$  and  $\cos \theta$  between  $\theta=0$  and  $\theta=\pi$ . Find the points where the graphs intersect.  
 [ *C. U. 1936, '46* ]

5. Construct the graphs of  $\tan x$  and  $\cos x$  between 0 and  $\frac{1}{2}\pi$  for  $x$ , making a tabulation of the values of  $y$  dividing the interval into 9 equal parts.

If  $\tan x = \cos x$ , find approximately the value of  $x$  from the above two graphs. [ *C. U. 1943* ]

6. Obtain graphically a solution of the equation  $\tan x = 1$ , between  $x=0$  and  $x=\frac{1}{2}\pi$ . [ *C. U. 1937* ]

[ *Draw the graphs of  $y = \tan x$  and  $y = 1$ .* ]



7. Draw the graph of  $\cos x - \sin 2x$  for values of  $x$  lying between  $0^\circ$  and  $90^\circ$  and hence obtain the least value of  $\cos x - \sin 2x$  in this range.

8. Solve graphically the equations :

(i)  $x - \tan x = 0$ , between  $x = 0$  and  $x = \frac{1}{2}\pi$ .

[ C. U. 1945 ]

(ii)  $5 \sin \theta + 2 \cos \theta = 5$ , between  $\theta = 0^\circ$  and  $\theta = 270^\circ$ .

[ Draw the graphs of  $y = 5 \sin \theta + 2 \cos \theta$  and  $y = 5$  and find the common points. ]

[ C. U. 1947 ]

(iii)  $\cot \theta - \tan \theta = 2$ , between  $\theta = 0$  and  $\theta = \pi$ .

[ C. U. 1949 ]

(iv)  $\operatorname{cosec} x = \cot x + \sqrt{3}$ , between  $x = 0$  and  $x = \pi$ .

(v)  $\cos x = \sin 2x + \frac{1}{2}$ , between  $x = -\frac{1}{2}\pi$  and  $x = +\frac{1}{2}\pi$ .

(vi)  $5 - \tan x = 2x$ , between  $0$  and  $2\pi$ .

(vii)  $2 \sin x + x - 3 = 0$ .

(viii)  $x^2 = \cos x$ .

(ix)  $x = \cos^2 x$ .

[ Draw the graphs of  $y = \cos 2x$  and  $y = 2x - 1$ . ]

9. Represent by a graph the displacement given by

$$s = 2 \sin t + \sin 3t.$$

10. Show graphically that the equation  $2 \sin x + \cos 2x = \frac{1}{2}x$  has only three real roots.

11. Sketch the graphs :

$y = x$ ,  $y = \sin x$ ,  $y = \tan x$ , in  $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$ . From the nature of graphs near the origin, can you suggest any relation among them at the origin ?

[ C. U. 1952 ]

## CHAPTER XVII

### MISCELLANEOUS THEOREMS AND EXAMPLES

#### Sec. A

#### HARDER PROBLEMS ON HEIGHTS AND DISTANCES

**112.** Some simple practical applications of Trigonometry, dealing with easy problems on determination of heights and distances, have already been discussed in Chapter V. The problems in the present section are of a more general character, requiring for their solutions, the general relations between the sides and angles of a triangle, as also some geometrical skill.

**113.** *To find the height and the distance of an inaccessible object standing on a horizontal plane.*

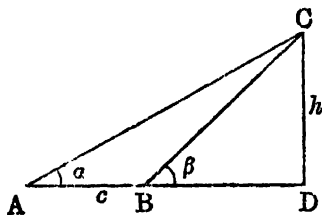


Fig. (i)

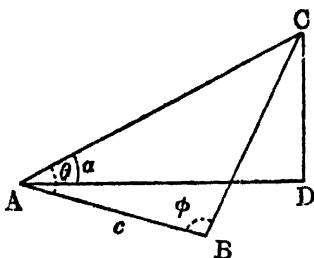


Fig. (ii)

Let  $CD$  be the object, which is observed from a point  $A$  on a horizontal ground,  $\alpha$  being the observed elevation of its top  $C$ . Let  $h$  be the required height  $CD$  and  $d$  the required distance  $AD$  of the object from  $A$ .

*Case I.* If possible, measure off any suitable distance  $AB (=c)$  from  $A$  directly towards the object, and from  $B$  let the observed elevation of  $C$  be  $\beta$ .

Then, from fig. (i),

$$\begin{aligned} c &= AD - BD = h \cot \alpha - h \cot \beta \\ &= h \left( \frac{\cos \alpha}{\sin \alpha} - \frac{\cos \beta}{\sin \beta} \right) = \frac{h \sin (\beta - \alpha)}{\sin \alpha \sin \beta} \end{aligned}$$

$$\therefore h = c \sin \alpha \sin \beta \operatorname{cosec} (\beta - \alpha).$$

$$\text{Also } d = AD = h \cot \alpha = c \cos \alpha \sin \beta \operatorname{cosec} (\beta - \alpha).$$

**Note.** Each of the above expressions for  $h$  and  $d$  is in a suitable form for logarithmic computation.

*Case II.* If however it is not convenient to measure the length  $AB$  directly towards the object, we may proceed as follows.

Measure off the length  $AB (= c)$  in *any* suitable direction from  $A$ . From  $A$  let the observed elevation of  $C$  be  $\alpha$  as before. The angles  $CAB$  and  $CBA$  are also observed from  $A$  and  $B$  respectively. Let these be  $\theta$  and  $\phi$ .

We get from fig. (ii) in this case,

$$\text{in } \triangle ABC, \frac{AC}{\sin \phi} = \frac{AB}{\sin C},$$

$$\text{i.e., } AC = \frac{c}{\sin (180^\circ - \theta - \phi)} = \frac{c}{\sin (\theta + \phi)}.$$

$$\therefore AC = c \sin \phi \operatorname{cosec} (\theta + \phi).$$

$$\therefore h = AC \sin \alpha = c \sin \alpha \sin \phi \operatorname{cosec} (\theta + \phi),$$

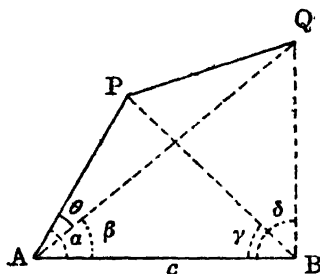
$$\text{and } d = AD = AC \cos \alpha = c \cos \alpha \sin \phi \operatorname{cosec} (\theta + \phi).$$

**Note.** Here also, the expressions for  $h$  and  $d$  are suitable for calculation by logarithm.

**114.** To find the distance between two visible but inaccessible objects.

Let  $P$  and  $Q$  be the objects whose distance apart is required.

Take two suitable points  $A$  and  $B$  for observation, the distance between which is measured, say  $c$ .



At  $A$ , measure the angles  $PAQ$ ,  $PAB$ , and  $QAB$  (the second observation being unnecessary if all the four points  $P, A, B, Q$  are in the same plane, for in that case,  $\angle PAB = \angle PAQ + \angle QAB$ ). Let these be  $\theta$ ,  $\alpha$  and  $\beta$  respectively.

At  $B$ , measure the angles  $PBA$  and  $QBA$ , and let them be  $\gamma$  and  $\delta$ .

From  $\triangle PAB$ ,  $\frac{PA}{\sin \gamma} = \frac{c}{\sin (180^\circ - \alpha - \gamma)} = \frac{c}{\sin (\alpha + \gamma)}$ ,  
whence  $PA = c \sin \gamma \operatorname{cosec} (\alpha + \gamma)$ .

Similarly, from  $\triangle QAB$ ,

$$QA = c \sin \delta \operatorname{cosec} (\beta + \delta).$$

Lastly, from  $\triangle PAQ$ ,

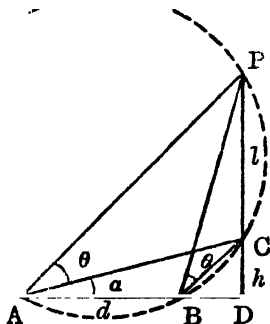
$$PQ^2 = PA^2 + QA^2 - 2PA \cdot QA \cdot \cos \theta.$$

Thus,  $PQ$  is determined.

115. Some more illustrative examples of harder problems on heights and distances are worked out below.

**Ex. 1.** A flagstaff is fixed on the top of a tower standing on a horizontal plane. An observer walking directly towards the foot of the tower, observes the angle subtended

by the flagstaff from two positions on his path to be the same namely  $\theta$ . The distance between these two positions is  $d$ , and the angle subtended by the tower at his first position is  $\alpha$ . Find the height of the tower and the length of the flagstaff.



Let  $CD$  be the tower,  $PC$  the flagstaff, whose heights required are  $h$  and  $l$  respectively.  $A$  and  $B$  are the points of observation.

$\therefore \angle PAC = \angle PBC = \theta$ , the points  $P, A, B, C$  are concyclic,

$$\therefore \angle CBD = \angle APC = 90^\circ - \angle PAD = 90^\circ - (\theta + \alpha).$$

$$\text{Now, } d = AD - BD = h \cot \alpha - h \cot (\angle CBD)$$

$$= h \{ \cot \alpha - \tan (\theta + \alpha) \}$$

$$= h \left\{ \frac{\cos \alpha}{\sin \alpha} - \frac{\sin (\theta + \alpha)}{\cos (\theta + \alpha)} \right\} = h \frac{\cos (\theta + 2\alpha)}{\sin \alpha \cos (\theta + \alpha)}.$$

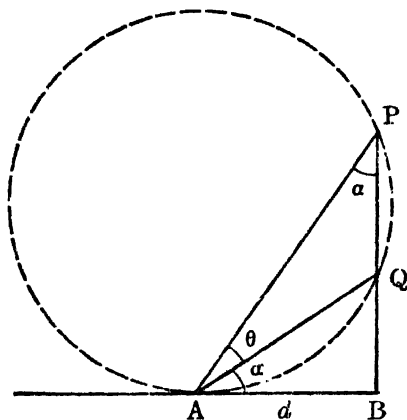
$$\therefore h = d \sin \alpha \cos (\theta + \alpha) \sec (\theta + 2\alpha).$$

Again, from  $\triangle APC$ ,

$$\frac{l}{\sin \theta} = \frac{AC}{\sin APC} = \frac{h}{\sin \alpha \cos (\theta + \alpha)} = \frac{d}{\cos (\theta + 2\alpha)}.$$

$$\therefore l = d \sin \theta \sec (\theta + 2\alpha).$$

**Ex. 2.** A man walking towards a building, on which a flagstaff is fixed, observes the angle subtended by the flagstaff to be greatest, when he is at a distance  $d$  from the building. If  $\theta$  be the observed greatest angle, find the length of the flagstaff and the height of the building.



Let  $QB$  be the building, and  $PQ$  the flagstaff. It is easily seen from Geometry that the point of contact  $A$  of a circle through  $P$  and  $Q$  touching the path of the observer on the ground, is the point at which the angle subtended by  $PQ$  is the greatest.

Thus,  $\angle QAB = \angle APQ = \alpha$  say.

Then,  $\angle PAB + \angle APB = 90^\circ$

$$\text{or, } \theta + 2\alpha = 90^\circ. \quad \dots (i)$$

Now,  $PQ = PB - QB = d \tan (\theta + \alpha) - d \tan \alpha$

$$= d \left\{ \frac{\sin (\theta + \alpha)}{\cos (\theta + \alpha)} - \frac{\sin \alpha}{\cos \alpha} \right\}$$

$$= d \frac{\sin \theta}{\cos (\theta + \alpha) \cos \alpha} = \frac{2d \sin \theta}{\cos (\theta + 2\alpha) + \cos \theta}$$

$$= 2d \tan \theta. \quad [\text{from (i)}]$$

$$\text{Also, } QB = d \tan \alpha = d \tan \left( \frac{\pi}{4} - \frac{\theta}{2} \right).$$

**Ex. 3.** The angle of elevation of a light at the top of a distant tower from a point 12 ft. above a lake is  $24^\circ 55'$ , and the angle of depression of its reflection in the lake is  $35^\circ 5'$ . Find the height of the tower correct to two decimal places, having given

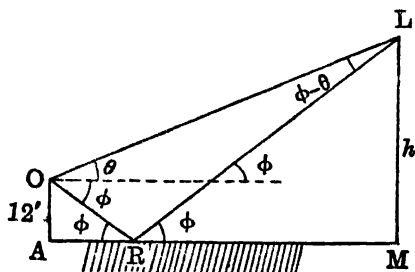
$$\log 2 = .30103,$$

$$\log 3 = .47712$$

$$\log 588 = 2.76938,$$

$$\log 589 = 2.77012$$

$$L \sin 10^\circ 10' = 9.24677.$$



Let  $L$  be the light at the top of the tower  $LM$ ,  $LRO$  the ray from  $L$ , which reflected in the lake at  $R$ , reaches the observer  $O$ , so that  $OR$  is the direction in which the reflexion is seen, and thus from the laws of reflexion,  $\angle ORA = \angle LRM = \phi$  (say) which is evidently equal to the angle of depression of the reflexion, i.e.,  $35^\circ 5'$ .

Let  $\theta$  denote the angle of elevation of  $L$  from  $O$ , i.e.,  $24^\circ 55'$ .

Now, from the figure, in  $\triangle ORL$ ,

$$\frac{RL}{\sin(\theta + \phi)} = \frac{OR}{\sin(\phi - \theta)} = \frac{12}{\sin \phi \sin(\phi - \theta)} \text{ ft.}$$

$$h = LM = RL \sin \phi = 12 \frac{\sin(\theta + \phi)}{\sin(\phi - \theta)} = 12 \frac{\sin 60^\circ}{\sin(10^\circ 10')}$$

$$= \frac{6\sqrt{3}}{\sin(10^\circ 10')} = \frac{2.3^{\frac{3}{2}}}{\sin(10^\circ 10')}.$$

$$\begin{aligned}
 \text{Hence, } \log h &= \log (2.3^{\frac{8}{3}}) - \log \sin (10^\circ 10') \\
 &= \log 2 + \frac{8}{3} \log 3 + 10 - I \sin (10^\circ 10') \\
 &= .30103 + \frac{8}{3} (.47712) + 10 - 9.24677 \\
 &= 1.76994.
 \end{aligned}$$

From the given data, it is seen that

$\log h$  lies between  $\log 58.8$  and  $\log 58.9$ .

$$\begin{aligned}
 \text{Hence, if } h = 58.8 + x, \text{ diff. for } .1 &= 1.77012 - 1.76938 \\
 &= .00074,
 \end{aligned}$$

$$\text{and diff. for } x = 1.76994 - 1.76938 = .00056.$$

$\therefore$  by the theory of proportional parts,

$$\frac{x}{.1} = \frac{.56}{.74} = .75; \quad \therefore x = .075 = .08 \text{ approximately.}$$

Thus,  $h = 58.88$  ft.

### Examples XVII(a)

1. The angles of elevation of the top of a plam tree standing on horizontal ground, observed from two points  $A$  and  $B$ , distant 40 and 30 feet from the foot, and in the same straight line with it are found to be complementary. Prove that the height of the tree is nearly 35 feet, and that the angle subtended at the top of the tree by the line  $AB$  is  $\sin^{-1} \frac{1}{2}$ .

2. The angles of elevation of an aeroplane from two places one mile apart and from a point half way between them are found to be  $60^\circ$ ,  $30^\circ$  and  $45^\circ$  respectively. Show that the height of the aeroplane  $440\sqrt{6}$  yards.

3. A building with ten storeys, each of equal height  $x$  ft., stands on one side of a wide street, and from a point on



the other side of the street directly opposite to the building, it is observed that the three uppermost storeys together and the two lowest storeys together subtend equal angles. Show that the width of the street is  $x\sqrt{140}$  ft.

4. A two-storeyed building has the height of its lower storey 12 ft. and that of the upper storey 13 ft. Find at what distance the two storeys subtend equal angles to an observer's eye at a height 5 feet from the ground.

5. A vertical rod is erected in a horizontal rectangular field  $ABCD$ . The angular elevation of its top from  $A, B, C, D$  are  $\alpha, \beta, \gamma, \delta$ . Show that

$$\cot^2 \alpha - \cot^2 \beta = \cot^2 \delta - \cot^2 \gamma.$$

6. The angles of elevation of a bird flying in a horizontal straight line, from a fixed point at four successive observations are  $\alpha, \beta, \gamma, \delta$ , the observations being taken at equal intervals of time. Assuming that the speed of the bird is uniform, show that

$$\cot^2 \alpha - \cot^2 \delta = 3(\cot^2 \beta - \cot^2 \gamma).$$

7. A man on a hill observes that three towers on a horizontal plane subtend equal angles at his eye and that the angles of depression of their bases are  $\alpha, \beta, \gamma$ . If  $a, b, c$  are the heights of the towers, prove that

$$\frac{\sin(\beta - \gamma)}{a \sin \alpha} + \frac{\sin(\gamma - \alpha)}{b \sin \beta} + \frac{\sin(\alpha - \beta)}{c \sin \gamma} = 0.$$

8. A gun is fired from a fort  $F$  at a distance  $d$  from a station  $O$ , and from two stations  $A$  and  $B$  in a straight line with  $O$  and distant  $a$  and  $b$  respectively from  $O$ , the intervals between seeing the flash and hearing the report are  $t$  and  $t'$ . Show that the velocity of sound is

$$\sqrt{\frac{(d^2 - ab)(a - b)}{at'^2 - bt'^2}}.$$

9. A person observes the elevation of the top of a telegraph post which is E. S. E. of him to be  $45^\circ$ , and at noon, the extremity of its shadow is to the N. E. of him; if the length of the shadow be  $x$ , show that the height of the post is  $x\sqrt{2 - \sqrt{2}}$ .

10. A straight tree on the horizontal ground leans towards the North ; at two points due South and distant  $a, b$  respectively from the foot, the angular elevations of the top of the tree are  $\alpha$  and  $\beta$ . Show that the inclination of the tree to the horizon is

$$\tan^{-1} \left( \frac{a - b}{a \cot \beta - \cot \alpha} \right).$$

11. An observer on a carriage moving with a speed  $V$  along a straight road observes in one position that two distant trees are in the same line with him which is inclined at an angle  $\theta$  to the road. After a time  $t$ , he observes that the trees subtend their greatest angle  $\phi$  ; show that the distance between the tree is

$$2Vt \sin \theta \sin \phi / (\cos \theta + \cos \phi).$$

12. A train travelling on one of two straight intersecting railways subtends at a certain station on the other line, angles  $\alpha$  and  $\beta$ , when the front of the first carriage and the end of the last carriage reach the junction respectively. Show that the angle of intersection of the two lines is

$$\tan^{-1} \frac{2 \sin \alpha \sin \beta}{\sin (\alpha + \beta)}.$$

13. Two vessels are sailing in parallel directions, and at one instant the bearing of one from the other  $\alpha^\circ$  N. of E. After an hour's sailing the bearing of the first from the second is  $\beta^\circ$  N. of E., and after another hour the bearing is  $\gamma^\circ$  N. of E. Show that the vessels are sailing in a direction  $\theta^\circ$  N. of E., where

$$\sin (\alpha - \theta) \sin (\gamma - \beta) = \sin (\beta - \alpha) \sin (\gamma - \theta).$$

14. A rod of given length can turn in a vertical plane passing through the sun, one end being fixed on the ground ; if the longest shadow it can cast is  $3\frac{1}{2}$  times the length of the rod, calculate the altitude of the sun, having given

$$\log 3 = \cdot 47712, L \cos 72^\circ 32' 30'' = 9 \cdot 47712.$$

15. A ship sailing N. E. is, at a particular moment, in a line with two light-houses, one of which is situated 5 miles

due N. of the other. In 3 minutes and also in 21 minutes the light-houses are found to subtend a right angle at the ship. Prove that the ship is sailing at the rate of 10 miles an hour, and that the light-houses subtend their greatest angle at the ship at the end of  $3\sqrt{7}$  minutes.

16. A parachute was observed in the N. E. at the elevation  $45^\circ$ ; ten minutes afterwards it was found to be due N. at an elevation  $22\frac{1}{2}^\circ$ . The parachute was descending at the rate of 6 miles per hour, and was all along drifted uniformly towards the West by the wind. Show that wind blows at the rate of 6 miles per hour.

17. A person wishing to determine the height of a distant temple observes the elevation of its top from a point on the horizontal ground through its base to be  $30^\circ$ . On walking a distance  $80\sqrt{3}$  ft. in a certain direction, he finds the elevation of the top to be the same as before, and then on walking a distance 80 ft. at right angles to the former direction, the elevation is found to be  $45^\circ$ . Show that the height of the temple is 80 ft.

18. The shadow of a telegraph post is observed to be half the known height of the post, and sometime afterwards it is equal to the known height; how much will the sun have gone down interval, given .

$$\log 2 = \cdot 30103, L \tan 63^\circ 26' = 10\cdot 3009994$$

$$\text{and diff. for } 1' = 3159.$$

19. The side of a hill faces due S., and is inclined to the horizon at an angle  $\alpha$ . A straight railway upon it is inclined at an angle  $\beta$  to the horizon; show that the bearing of the railway is

$$\cos^{-1}(\cot \alpha \tan \beta) \text{ E. of N.}$$

20. A spherical time-ball of diameter  $d$  at the top of a tower subtends an angle  $2\alpha$  at a point on the ground from which the elevation of its centre is  $\theta$ ; prove that the height of the centre of the ball above the ground is  $\frac{1}{2}d \sin \theta \operatorname{cosec} \alpha$ .

## Sec. B—SUMMATION OF FINITE SERIES

## 116. Method of Difference.

When the  $r$ th term of a trigonometrical series can be expressed as the difference of two quantities, one of which is the same function of  $r$  as the other is of  $(r+1)$ , the sum of the series may be readily found as illustrated in the Examples 1 and 2 below.

Ex. 1. Find the sum of the series :

$$(i) \operatorname{cosec} \theta + \operatorname{cosec} 2\theta + \operatorname{cosec} 2^2\theta + \cdots + \operatorname{cosec} 2^{n-1}\theta.$$

$$(ii) \frac{\sin x}{\sin 2x \sin 3x} + \frac{\sin x}{\sin 3x \sin 4x} + \frac{\sin x}{\sin 4x \sin 5x} \\ + \cdots \text{to } n \text{ terms.}$$

$$(i) \text{ We have } \operatorname{cosec} \theta = \frac{1}{\sin \theta} = \frac{\sin \frac{1}{2}\theta}{\sin \frac{1}{2}\theta \sin \theta} \\ = \frac{\sin (\theta - \frac{1}{2}\theta)}{\sin \frac{1}{2}\theta \sin \theta} \\ = \frac{\sin \theta \cos \frac{1}{2}\theta - \cos \theta \sin \frac{1}{2}\theta}{\sin \frac{1}{2}\theta \sin \theta} \\ = \cot \frac{1}{2}\theta - \cot \theta.$$

$$\text{Thus,} \quad \operatorname{cosec} \theta = \cot \frac{1}{2}\theta - \cot \theta.$$

$$\text{Similarly,} \quad \operatorname{cosec} 2\theta = \cot \theta - \cot 2\theta.$$

$$\operatorname{cosec} 2^2\theta = \cot 2\theta - \cot 2^2\theta.$$

.....

$$\operatorname{cosec} 2^{n-1}\theta = \cot 2^{n-2}\theta - \cot 2^{n-1}\theta.$$

$\therefore$  by addition, the required sum

$$= \cot \frac{1}{2}\theta - \cot 2^{n-1}\theta.$$

(ii) Here,  $r$ th term

$$\begin{aligned}
 &= \frac{\sin x}{\sin (r+1) x \sin (r+2) x} \\
 &= \frac{\sin \{(r+2)-(r+1)\} x}{\sin (r+1) x \sin (r+2) x} \\
 &= \frac{\sin (r+2) x \cos (r+1) x - \cos (r+2) x \sin (r+1) x}{\sin (r+1) x \sin (r+2) x} \\
 &= \cot (r+1) x - \cot (r+2) x.
 \end{aligned}$$

Putting  $r=1, 2, 3, \dots, n$  and adding, the sum of the required series would be found to be

$$\cot 2x - \cot (n+2) x.$$

**Ex. 2.** Find the sum of the series

$$\begin{aligned}
 \tan^{-1} \frac{x}{1+1.2x^2} + \tan^{-1} \frac{x}{1+2.3x^2} + \dots \\
 + \tan^{-1} \frac{x}{1+n(n+1)x^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{Here, } r\text{th term} &= \tan^{-1} \frac{x}{1+r(r+1)x^2} \\
 &= \tan^{-1} \frac{(r+1)x - rx}{1+(r+1)x \cdot rx} \\
 &= \tan^{-1} (x+1)x - \tan^{-1} rx.
 \end{aligned}$$

$\therefore$  putting  $r=1, 2, 3, \dots, n$ , we have

$$\begin{aligned}
 \tan^{-1} \frac{x}{1+1.2x^2} &= \tan^{-1} 2x - \tan^{-1} x \\
 \tan^{-1} \frac{x}{1+2.3x^2} &= \tan^{-1} 3x - \tan^{-1} 2x \\
 &\dots\dots\dots \\
 \tan^{-1} \frac{x}{1+n(n+1)x^2} &= \tan^{-1} (n+1)x - \tan^{-1} nx.
 \end{aligned}$$

by addition, the required sum

$$= \tan^{-1} (n+1)x - \tan^{-1} x.$$

117. Sometimes the  $r$ th term of a series, when multiplied by a factor, can be expressed as the difference of two quantities one of which is the same function of  $r$  as the other is of  $(r+1)$ . It is illustrated in the following two cases.

(I) Sum of sines of  $n$  angles in A.P.

Let the angles in A.P. be  $\alpha, \alpha + \beta, \alpha + 2\beta, \dots \{\alpha + (n-1)\beta\}$ , the first term being  $\alpha$ , and the common difference,  $\beta$ .

Let  $S$  denote the sum of the series,

$$\sin \alpha + \sin (\alpha + \beta) + \sin (\alpha + 2\beta) + \dots + \sin \{\alpha + (n-1)\beta\}.$$

Multiplying each term of the above series by

$2 \sin (\text{half difference})$  i.e., by  $2 \sin \frac{1}{2}\beta$ , we have

$$\begin{aligned} 2 \sin \alpha \sin \frac{1}{2}\beta &= \cos \left(\alpha - \frac{1}{2}\beta\right) - \cos \left(\alpha + \frac{1}{2}\beta\right) \\ 2 \sin (\alpha + \beta) \sin \frac{1}{2}\beta &= \cos \left(\alpha + \frac{1}{2}\beta\right) - \cos \left(\alpha + \frac{3}{2}\beta\right) \\ 2 \sin (\alpha + 2\beta) \sin \frac{1}{2}\beta &= \cos \left(\alpha + \frac{3}{2}\beta\right) - \cos \left(\alpha + \frac{5}{2}\beta\right). \\ &\dots\dots\dots \end{aligned}$$

$$\begin{aligned} 2 \sin \{\alpha + (n-1)\beta\} \sin \frac{1}{2}\beta \\ = \cos \left(\alpha + \frac{2n-3}{2}\beta\right) - \cos \left(\alpha + \frac{2n-1}{2}\beta\right). \end{aligned}$$

Adding vertically, we have

$$\begin{aligned} 2 \sin \frac{1}{2}\beta \cdot S &= \cos \left(\alpha - \frac{\beta}{2}\right) - \cos \left(\alpha + \frac{(2n-1)}{2}\beta\right) \\ &\quad - 2 \sin \left(\alpha + \frac{n-1}{2}\beta\right) \sin \frac{n\beta}{2}. \\ S &= \frac{\sin \frac{n\beta}{2}}{\sin \frac{\beta}{2}} \sin \left(\alpha + \frac{n-1}{2}\beta\right). \end{aligned}$$

Cor. Putting  $\beta = a$ , we get

$$\begin{aligned} \sin \alpha + \sin 2\alpha + \sin 3\alpha + \dots + \sin n\alpha \\ = \frac{\sin \frac{na}{2}}{\sin \frac{a}{2}} \sin \frac{n+1}{2} a \end{aligned}$$

Note. The sum of the cosine series may be deduced from that of the sine series by writing  $a + \frac{\pi}{2}$  for  $a$ .

As an aid to memory, the two formulæ of this article may be expressed in language as follows :

$$\text{since, } a + \frac{n-1}{2}\beta = \frac{a + a + (n-1)\beta}{2},$$

∴ **Sum of sines of  $n$  angles in A.P.**

$$= \frac{\sin \frac{n \cdot \text{diff.}}{2}}{\sin \frac{\text{diff.}}{2}} \sin \frac{\text{first angle} + \text{last angle}}{2}.$$

**Sum of cosines of  $n$  angles in A.P.**

$$= \frac{\sin \frac{n \cdot \text{diff.}}{2}}{\sin \frac{\text{diff.}}{2}} \cos \frac{\text{first angle} + \text{last angle}}{2}.$$

**Note.** From the above formulæ, it is clear that if  $\sin \frac{n\beta}{2} = 0$ , then the sum of the sine series as also the sum of the cosine series is zero.

Now, if  $\sin \frac{n\beta}{2} = 0$ , then  $\frac{n\beta}{2} = k\pi$ , or  $\beta = \frac{2k\pi}{n}$ , where  $k$  is an integer.

Thus, the sum of the sines and the sum of the cosines of  $n$  angles in A.P. are each equal to zero when the common difference of the angles is an even multiple of  $\frac{\pi}{n}$ .

**Ex. 1.** Find the sum of  $n$  terms of the series

$$\sin a - \sin(a + \beta) + \sin(a + 2\beta) - \dots$$

Since,  $\sin(\pi + \theta) = -\sin \theta$  ;  $\sin(2\pi + \theta) = \sin \theta$  etc.

∴ the series is equal to

$$\sin a + \sin(\pi + a + \beta) + \sin(2\pi + a + 2\beta) + \dots,$$

i.e., equal to a series in which the common difference of the angles is  $\beta + \pi$  and the last angle is  $a + (n-1)(\beta + \pi)$ .

$$S = \frac{\sin \frac{n(\beta + \pi)}{2}}{\sin \frac{\beta + \pi}{2}} \sin \left\{ a + \frac{(n-1)(\beta + \pi)}{2} \right\}.$$



**Ex. 2.** Find the sum of the series

$$\sin^2 \theta + \sin^2 2\theta + \sin^2 3\theta + \dots + \sin^2 n\theta.$$

Since,  $\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$ ,  $\sin^2 2\theta = \frac{1}{2}(1 - \cos 4\theta)$ , etc.

$\therefore$  the given series

$$\begin{aligned} &= \frac{1}{2}(1 - \cos 2\theta) + \frac{1}{2}(1 - \cos 4\theta) + \dots + \frac{1}{2}(1 - \cos 2n\theta) \\ &= \frac{n}{2} - \frac{1}{2}(\cos 2\theta + \cos 4\theta + \dots + \cos 2n\theta) \\ &= \frac{n}{2} - \frac{1}{2} \frac{\sin n\theta}{\sin \theta} \cos (n+1)\theta. \quad [\text{by Art. 117}] \end{aligned}$$

**Ex. 3.** Sum the series

$$\begin{aligned} &\cos \alpha + 2 \cos (\alpha + \beta) + 3 \cos (\alpha + 2\beta) + \dots \\ &\dots + n \cos \{\alpha + (n-1)\beta\}. \end{aligned}$$

Let  $u_r$  denote the  $r$ th term and  $S$  denote the sum of the given series.

$$\begin{aligned} \text{Now, } 2 \cos \beta \cdot u_r &= 2 \cos \beta \cdot r \cos \{\alpha + (r-1)\beta\} \\ &= r [\cos (\alpha + r\beta) + \cos \{\alpha + (r-2)\beta\}]. \end{aligned}$$

$\therefore$  putting  $r = 1, 2, 3, \dots, n$  and adding together,

we get  $2 \cos \beta \cdot S$ .

Now, subtract  $2 \cos \beta \cdot S$  from  $2S$ ; then

$$\begin{aligned} 2S(1 - \cos \beta) &= (n+1) \cos \{\alpha + (n-1)\beta\} \\ &\quad - \cos (\alpha - \beta) - n \cos (\alpha + n\beta). \end{aligned}$$

Then, dividing by  $2(1 - \cos \beta)$ ,  $S$ , the sum of the required series would be obtained.

**Note.** Similarly the sum of the series

$$\sin \alpha + 2 \sin (\alpha + \beta) + 3 \sin (\alpha + 2\beta) + \dots + n \sin \{\alpha + (n-1)\beta\}$$

would be obtained.

## Examples XVII(b)

Sum the following series to  $n$  terms (*Ex. 1 to 10*) :

1.  $\sin a + \sin \left( a - \frac{\pi}{n} \right) + \sin \left( a - \frac{2\pi}{n} \right) + \dots$
2.  $\cos a + \cos \left( a + \frac{2\pi}{n} \right) + \cos \left( a + \frac{4\pi}{n} \right) + \dots$
3.  $\sin a - \sin 2a + \sin 3a - \dots$
4.  $\cos^2 \theta + \cos^2 2\theta + \cos^2 3\theta + \dots$
5.  $\sin^3 a + \sin^3 3a + \sin^3 5a + \dots$
6.  $\sin^3 \theta - \sin^3 2\theta + \sin^3 3\theta - \sin^3 4\theta + \dots$
7.  $\sin^4 a + \sin^4 2a + \sin^4 3a + \dots$
8.  $\cos \theta - \sin 2\theta - \cos 3\theta + \sin 4\theta + \cos 5\theta - \sin 6\theta - \dots$
9.  $\sin a \sin 2a + \sin 2a \sin 3a + \sin 3a \sin 4a + \dots$
10.  $\cos a \cos 3a + \cos 3a \cos 5a + \cos 5a \cos 7a + \dots$

Find the sum of the following series (*Ex. 11 to 14*) :

11.  $\cos \frac{\pi}{19} + \cos \frac{3\pi}{19} + \cos \frac{5\pi}{19} + \dots + \cos \frac{17\pi}{19}$
12.  $\sin 5^\circ + \sin 77^\circ + \sin 149^\circ + \dots + \sin 293^\circ$
13.  $\sin \frac{2\pi}{n} + \sin \frac{4\pi}{n} + \sin \frac{6\pi}{n} + \dots + \sin \frac{2n\pi}{n}$
14.  $\sin na + \sin (n-1)a + \sin (n-2)a + \dots$  to  $2n$  terms.
15. Prove that

$$(i) \frac{\sin \theta + \sin 3\theta + \sin 5\theta + \dots \text{ to } n \text{ terms}}{\cos \theta + \cos 3\theta + \cos 5\theta + \dots \text{ to } n \text{ terms}} = \tan n\theta.$$

$$(ii) \sin^2 a + \sin^2 \left( a + \frac{2\pi}{n} \right) + \sin^2 \left( a + \frac{4\pi}{n} \right) + \dots \text{ to } n \text{ terms} \\ = \frac{1}{2}n.$$

Sum to  $n$  terms (*Ex. 16 to 28*) :

$$16. \sec a \sec 2a + \sec 2a \sec 3a + \sec 3a \sec 4a + \dots$$

$$17. \frac{1}{\sin \theta \sin 2\theta} + \frac{1}{\sin 2\theta \sin 3\theta} + \frac{1}{\sin 3\theta \sin 4\theta} + \dots$$

$$18. \frac{1}{\cos a + \cos 3a} + \frac{1}{\cos a + \cos 5a} + \frac{1}{\cos a + \cos 7a} + \dots$$

$$19. \cot \theta \cot 2\theta + \cot 2\theta \cot 3\theta + \cot 3\theta \cot 4\theta + \dots$$

$$20. \tan a + 2 \tan 2a + 4 \tan 4a + 8 \tan 8a + \dots$$

$$[ \tan a = \cot a - 2 \cot 2a ]$$

$$21. \sin 2\theta \sin^2 \frac{2\theta}{2} + \sin 3\theta \sin^2 \frac{3\theta}{2} + \sin 4\theta \sin^2 \frac{4\theta}{2} + \dots$$

$$22. \frac{\sin x}{\cos 3x} + \frac{\sin 3x}{\cos 3^2 x} + \frac{\sin 3^2 x}{\cos 3^3 x} + \dots$$

$$[ \text{1st term} = \frac{1}{2} (\tan 3x - \tan x) ]$$

$$23. \tan^{-1} \frac{1}{1+1+1^2} + \tan^{-1} \frac{1}{1+2+2^2} \\ + \tan^{-1} \frac{1}{1+3+3^2} + \dots$$

$$24. \tan^{-1} \frac{2}{1+1.3} + \tan^{-1} \frac{2}{1+3.5} + \tan^{-1} \frac{2}{1+5.7} + \dots$$

$$25. \cot^{-1} (2.1^2) + \cot^{-1} (2.2^2) + \cot^{-1} (2.3^2) + \dots$$

$$26. \tan x + \frac{1}{2} \tan \frac{1}{2} x + \frac{1}{2^2} \tan \frac{1}{2^2} x + \dots$$

$$27. \cos x \cos 2x \cos 3x + \cos 2x \cos 3x \cos 4x + \dots$$

$$28. \cos \theta + 2 \cos 2\theta + 3 \cos 3\theta + \dots + n \cos n\theta.$$

$$29. \text{Find the sum of the series :}$$

$$(i) \sin a + \sin 2a + \sin 3a + \dots \text{ to } n \text{ terms}$$

$$\text{and } (ii) \sin a + \sin 3a + \sin 5a + \dots \text{ to } n \text{ terms}$$

and hence deduce respectively the sums of the series

(a)  $1 + 2 + 3 + \dots$  to  $n$  terms

and (b)  $1 + 3 + 5 + \dots$  to  $n$  terms.

30. Sum the series

$\tan x \tan 2x + \tan 2x \tan 3x + \dots + \tan nx \tan (n+1)x$   
and hence deduce the sum of the series

$$1.2 + 2.3 + \dots + n(n+1).$$

31. If  $\beta$  be the exterior angle of a regular polygon of  $n$  sides, show that

(i)  $\sin a + \sin (a + \beta) + \sin (a + 2\beta) + \dots$  to  $n$  terms  $= 0$ .

(ii)  $\cos a + \cos (a + \beta) + \cos (a + 2\beta) + \dots$  to  $n$  terms  $= 0$ .

32. A regular polygon of  $n$  sides is inscribed in a circle of radius  $a$ ; prove that

(i) the sum of the lengths of the perpendiculars drawn from the angular points upon any diameter is zero;

(ii) the sum of the lengths of the lines joining any vertex to each of the other vertices is  $2a \cot \frac{\pi}{2n}$ .

## Sec. C—ELIMINATION

118. The elimination of trigonometrical functions from given equations is a very important and common mathematical problem. There are no set rules to effect the elimination. The form of the equations will often suggest special methods, and in addition to the usual algebraical artifices, we shall always have at our disposal the identical relations existing among the trigonometrical functions.

The following examples will illustrate some useful methods of elimination.

**Ex. 1.** *Eliminate  $\theta$  from the equations*

$$a \cos \theta + b \sin \theta + c = 0$$

$$a' \cos \theta + b' \sin \theta + c' = 0.$$

From the given equations, we have by cross-multiplication,

$$\frac{\cos \theta}{bc' - b'c} = \frac{\sin \theta}{ca' - c'a} = \frac{1}{ab' - a'b}.$$

$$\therefore \cos \theta = \frac{bc' - b'c}{ab' - a'b}, \text{ and } \sin \theta = \frac{ca' - c'a}{ab' - a'b}.$$

Squaring and adding, we get

$$(bc' - b'c)^2 + (ca' - c'a)^2 = (ab' - a'b)^2$$

as the required eliminant.

**Ex. 2.** *Eliminate  $\theta$  from the equations*

$$x \sin \theta + y \cos \theta = 2a \sin 2\theta$$

$$x \cos \theta - y \sin \theta = a \cos 2\theta.$$

Solving as simultaneous equations in  $x$  and  $y$ , we have

$$x = a(\cos 2\theta \cos \theta + 2 \sin 2\theta \sin \theta)$$

$$= a[\cos (2\theta - \theta) + \sin 2\theta \sin \theta]$$

$$= a(\cos \theta + 2 \sin^2 \theta \cos \theta);$$

$$y = a(2 \sin 2\theta \cos \theta - \cos 2\theta \sin \theta)$$

$$= a(\sin \theta + \sin 2\theta \cos \theta) = a(\sin \theta + 2 \sin \theta \cos^2 \theta).$$

$$\therefore x + y = a(\sin \theta + \cos \theta)(1 + 2 \sin \theta \cos \theta)$$

$$= a(\sin \theta + \cos \theta)(\sin \theta + \cos \theta)^2 = a(\cos \theta + \sin \theta)^3.$$

Similarly,

$$x - y = a(\cos \theta - \sin \theta)(1 - 2 \sin \theta \cos \theta)$$

$$= a(\cos \theta - \sin \theta)^3.$$

$$\therefore a^{\frac{1}{3}}(\cos \theta + \sin \theta) = (x + y)^{\frac{1}{3}} \quad \dots \quad (i)$$

$$a^{\frac{1}{3}}(\cos \theta - \sin \theta) = (x - y)^{\frac{1}{3}} \quad \dots \quad (ii)$$

Hence, squaring and adding (i) and (ii), we have,

$$(x + y)^{\frac{2}{3}} + (x - y)^{\frac{2}{3}} = 2a^{\frac{2}{3}},$$

as the required eliminant.

**Ex. 3.** *Eliminate  $x$  and  $y$  from the equations*

$$a \sin^2 x + b \cos^2 x = c, \quad b \sin^2 y + a \cos^2 y = d,$$

$$a \tan x = b \tan y.$$

From the first equation, we have

$$a \sin^2 x + b \cos^2 x = c(\sin^2 x + \cos^2 x).$$

$$\therefore (a - c) \sin^2 x = (c - b) \cos^2 x.$$

$$\therefore \tan^2 x = \frac{c - b}{a - c}.$$

From the second equation, we have similarly

$$b \sin^2 y + a \cos^2 y = d(\sin^2 y + \cos^2 y).$$

$$\therefore \tan^2 y = \frac{d - a}{b - d}.$$

From the third equation,

$$a^2 \tan^2 x = b^2 \tan^2 y.$$

$$\therefore \frac{a^2(c - b)}{a - c} = \frac{b^2(d - a)}{b - d}.$$

This, when simplified, reduces to

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{c} + \frac{1}{d}, \text{ the required eliminant.}$$

## Examples XVII(c)

Eliminate  $\theta$  from the following pair of equations (*Ex. 1 to 15*) :

$$1. \quad \cot \theta (1 + \sin \theta) = 4a.$$

$$\cot \theta (1 - \sin \theta) = 4b.$$

$$2. \quad x = a \cos \theta + b \cos 2\theta.$$

$$y = a \sin \theta + b \sin 2\theta.$$

$$3. \quad x = \tan \theta + \tan 2\theta.$$

$$y = \cot \theta + \cot 2\theta.$$

$$4. \quad a \sin \theta + b \cos \theta = 1.$$

$$a \operatorname{cosec} \theta - b \sec \theta = 1.$$

$$5. \quad x = \sin \theta + \cos \theta \sin 2\theta$$

$$y = \cos \theta + \sin \theta \sin 2\theta.$$

$$6. \quad x + a = a(2 \cos \theta - \cos 2\theta)$$

$$y = a(2 \sin \theta - \sin 2\theta).$$

$$7. \quad x = 3 \sin \theta - \sin 3\theta$$

$$y = \cos 3\theta + 3 \cos \theta.$$

$$8. \quad x \cot \theta + \tan \theta$$

$$y = \sec \theta - \cos \theta.$$

$$9. \quad x \sin \theta - y \cos \theta = \sqrt{x^2 + y^2}$$

$$\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2} = \frac{1}{x^2 + y^2}.$$

$$10. \quad \frac{x}{a} = \cos \theta + \cos 2\theta$$

$$\frac{y}{b} = \sin \theta + \sin 2\theta.$$

$$11. \quad \frac{ax}{\cos \theta} - \frac{by}{\sin \theta} = a^2 - b^2$$

$$\frac{ax \sin \theta}{\cos^2 \theta} + \frac{by \cos \theta}{\sin^2 \theta} = 0.$$

$$12. \quad \frac{x}{a} \cos \theta - \frac{y}{b} \sin \theta = \cos 2\theta$$

$$\frac{x}{a} \sin \theta + \frac{y}{b} \cos \theta = 2 \sin 2\theta.$$

$$13. \quad x = \operatorname{cosec} \theta - \sin \theta$$

$$y = \sec \theta - \cos \theta.$$

$$14. \quad \sin \theta + \cos \theta = a$$

$$\sin^3 \theta + \cos^3 \theta = b.$$

$$15. \quad \frac{x}{a} \cos \theta + \frac{y}{b} \sin \theta = 1$$

$$x \sin \theta - y \cos \theta = (a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{\frac{1}{2}}.$$

Eliminate  $\theta$  and  $\phi$  from the following equations  
(Ex. 16 to 19):—

$$16. \quad \sin \theta + \sin \phi = x, \cos \theta + \cos \phi = y, \theta - \phi = a.$$

$$17. \quad \tan \theta + \tan \phi = a, \cot \theta + \cot \phi = b, \theta + \phi = a.$$

$$18. \quad a \sin^2 \theta + b \cos^2 \theta = a \cos^2 \phi + b \sin^2 \phi = 1,$$

$$a \tan \theta = b \tan \phi.$$

$$19. \quad \sin \theta + \sin \phi = a, \cos \theta + \cos \phi = b, \sin 2\theta + \sin 2\phi = 2c.$$

$$20. \quad \text{If } (a+b) \tan (\theta - \phi) = (a-b) \tan (\theta + \phi) \text{ and}$$

$$a \cos 2\phi + b \cos 2\theta = c, \text{ show that}$$

$$a^2 - b^2 + c^2 = 2ac \cos 2\phi.$$


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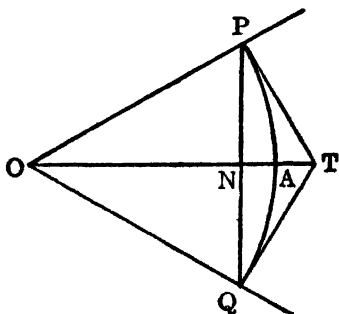


## APPENDIX

1. To prove that

$$\sin \theta < \theta < \tan \theta,$$

where  $\theta$  is the circular measure of any positive acute angle.



Let  $\angle AOP$  be a positive acute angle whose radian measure is  $\theta$ , and let  $\angle QOA$  be an equal angle on the other side of  $OA$ . With centre  $O$  and any radius, a circle is drawn cutting  $OP$ ,  $OA$ ,  $OQ$  at  $P$ ,  $A$ ,  $Q$  respectively.  $PQ$  is joined cutting  $OA$  at  $N$ . The triangles  $OPN$  and  $OQN$  are easily seen to be congruent, so that  $PN = QN$  and  $PNQ$  is perpendicular to  $OA$ . The tangent  $PT$  to the circle at  $P$  cuts  $OA$  at  $T$ , therefore  $\angle OPT$  is a right angle.  $TQ$  being joined, the triangles  $OPT$  and  $OQT$  are easily proved to be congruent, so that  $TP = TQ$ .

The figure is thus symmetrical about  $OA$ .

Then, from the figure,

$$\sin \theta = \frac{PN}{OP} = \frac{1}{2} \frac{PQ}{OP}$$

$$\theta = \frac{\text{arc } PA}{OP} = \frac{1}{2} \frac{\text{arc } PAQ}{OP}$$

$$\tan \theta = \frac{PT}{OP} = \frac{1}{2} \frac{PT + QT}{OP}.$$

Now, we may take it as axiomatic that the straight line  $PQ$  is less than the curved arc  $PAQ$ , and that the curved arc  $PAQ$  which always bends the same way, being within the triangle  $PTQ$ , is less than  $PT + TQ$ .

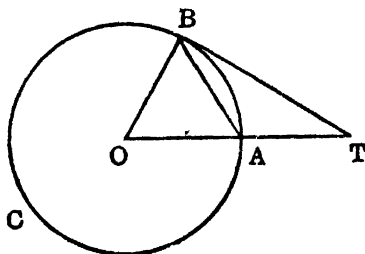
Hence, since  $PQ < \text{arc } PAQ < PT + TQ$ ,

we have, on dividing throughout by  $2OP$ ,

$$\sin \theta < \theta < \tan \theta.$$

*Alternative method :*

Let  $ABC$  be a circle whose centre is  $O$  and radius  $r$ ,



Let  $AOB = \theta$  radians.

Draw  $BT$  the tangent at  $B$  to meet  $OA$  produced at  $T$ ; then  $BT = r \tan \theta$ .

We know that if the angle of a sector of a circle of radius  $r$  is  $\theta$  radians, its area  $= \frac{1}{2}r^2\theta$ .

Now, from the figure it is obvious that

$$\triangle OAB < \text{sector } OAB < \triangle OBT.$$

$$\therefore \frac{1}{2}r^2 \sin \theta < \frac{1}{2}r^2\theta < \frac{1}{2}r \cdot r \tan \theta,$$

$$\text{i.e.,} \quad \sin \theta < \theta < \tan \theta.$$

**Cor.** If now  $\theta$  becomes infinitely small, we can prove

$$\text{Lt}_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1,$$

$$\lim_{\theta \rightarrow 0} \cos \theta = 1,$$

$$\text{and } \lim_{\theta \rightarrow 0} \frac{\tan \theta}{\theta} = 1.$$

For, since,  $\sin \theta < \theta < \tan \theta$ ,  
we get, by dividing by  $\sin \theta$ ,

$$1 < \frac{\theta}{\sin \theta} < \frac{1}{\cos \theta}.$$

This is true, however small  $\theta$  may be, provided it is positive. When  $\theta$  becomes infinitely small,  $OP$  and  $ON$  practically come into coincidence, so that

$$\cos \theta = \frac{ON}{OP} \text{ ultimately becomes } 1.$$

$$\text{Hence, } \lim_{\theta \rightarrow 0} \cos \theta = 1.$$

In that case  $\frac{1}{\cos \theta}$  also tends to the value 1. But  $\frac{\theta}{\sin \theta}$  always lies between 1 and  $\frac{1}{\cos \theta}$  which ultimately come into coincidence, and so  $\frac{\theta}{\sin \theta}$  also ultimately coincides with 1.

$$\text{Thus, } \frac{\sin \theta}{\theta} = 1 \text{ in the limit.}$$

Again, from

$$\sin \theta < \theta < \tan \theta,$$

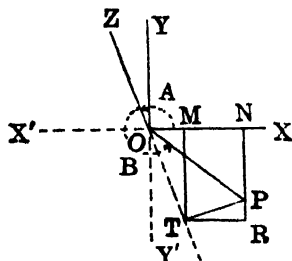
we get by dividing by  $\tan \theta$ ,

$$\cos \theta < \frac{\theta}{\tan \theta} < 1,$$

and as  $\theta \rightarrow 0$ ,  $\cos \theta \rightarrow 1$  and  $\frac{\theta}{\tan \theta}$  always lying between  $\cos \theta$  and 1 which come into coincidence,  $\frac{\theta}{\tan \theta} = 1$  in the limit, and so  $\lim_{\theta \rightarrow 0} \frac{\tan \theta}{\theta} = 1.$

Hence, the results.

2. *Formulae for  $\sin (A+B)$  and  $\cos (A+B)$  where  $A$  and  $B$  are of any magnitude. (Generalization of Art. 33)*



In Article 33, formulae for  $\sin (A+B)$  and  $\cos (A+B)$  were deduced geometrically with a figure in which  $A$  and  $B$  were acute and  $(A+B)$  less than  $90^\circ$ . We now prove them in a more general case.

Let a revolving line, starting from  $OX$ , trace out an angle  $XOZ = A$  and further trace out an angle  $ZOP = B$ , so that the total angle traced out is  $(A+B)$ . From any point  $P$  on the final position of the revolving line,  $PN$  and  $PT$  are drawn perpendiculars to  $OX$  and  $OZ$  (produced if necessary, as in the above figure), and from  $T$  perpendiculars  $TM$  and  $TR$  are drawn on  $OX$  and  $PN$  (produced if necessary).

In the figure above,  $\angle POT = B - 180^\circ$ , and since  $PN$  and  $PT$  are perpendiculars to  $OX$  and  $OZ$  respectively,  $\angle TPR = \angle TON = 180^\circ - \angle XOZ$ , i.e.,  $180^\circ - A$ .

In considering  $\sin (A+B)$  and  $\cos (A+B)$  from the triangle  $NOP$ , it is to be noted that  $PN$  is negative and  $ON$  and  $OP$  are positive.

If we consider only the positive magnitudes of the sides of the acute-angled triangles  $OTM$ ,  $PTR$  and  $OPT$ , then  $PN$  with its proper sign may be written as  $-(TM - PR)$ , and  $ON$  with its proper sign may be written as  $OM + TR$ .

Now, from the figure,

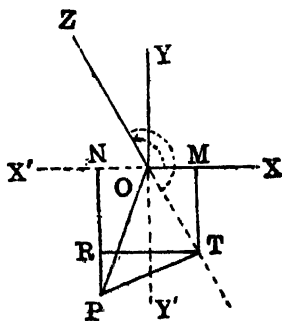
$$\sin (A+B) = \frac{PN}{OP} = -\frac{TM - PR}{OP}$$

$$\begin{aligned}
&= -\frac{TM}{OT} \frac{OT}{OP} + \frac{PR}{PT} \frac{PT}{OP} \\
&= -\sin TOM \cos POT + \cos TPR \sin POT \\
&= -\sin (180^\circ - A) \cos (B - 180^\circ) \\
&\quad + \cos (180^\circ - A) \sin (B - 180^\circ) \\
&= -\sin A (-\cos B) + (-\cos A)(-\sin B) \\
&= \sin A \cos B + \cos A \sin B.
\end{aligned}$$

Again,

$$\begin{aligned}
\cos (A + B) &= \frac{ON}{OP} = \frac{OM}{OP} + \frac{RT}{OP} \\
&= \frac{OM}{OT} \frac{OT}{OP} + \frac{RT}{PT} \frac{PT}{OP} \\
&= \cos TOM \cos POT + \sin TPR \sin POT \\
&= \cos (180^\circ - A) \cos (B - 180^\circ) \\
&\quad + \sin (180^\circ - A) \sin (B - 180^\circ) \\
&= (-\cos A)(-\cos B) + \sin A (-\sin B) \\
&= \cos A \cos B - \sin A \sin B.
\end{aligned}$$

3. *Formulae for  $\sin (A - B)$  and  $\cos (A - B)$  in a more general case. (Generalization of Art. 34)*



Here,  $XOZ$  measured counter-clockwise is  $A$  and  $ZOP$  measured clockwise has magnitude  $B$  so that  $XOP$  measured

clockwise is  $A - B$ . From  $P$ ,  $PN$  and  $PT$  are drawn perpendiculars on  $OX$  and  $OZ$  (produced in this figure), and from  $T$ ,  $TM$  and  $TR$  are drawn perpendiculars on  $OX$  and  $PN$ .

In the present figure, magnitudes of the acute angles  $TOM$  and  $POT$  are  $180^\circ - A$  and  $B - 180^\circ$  respectively, and noting that  $PNOT$  is a cyclic quadrilateral ( $\angle^s N$  and  $T$  being right angles),  $\angle RPT = \angle TOM = 180^\circ - A$  in magnitude.

Now, we see that in considering  $\sin(A - B)$  and  $\cos(A - B)$  from the triangle  $NOP$ ,  $PN$  and  $ON$  are of negative signs.

Hence,

$$\begin{aligned}\sin(A - B) &= \frac{PN}{OP} \\ &= -\frac{MT + PR}{OP},\end{aligned}$$

[where the magnitudes of  $MT$ ,  $PR$ , etc. only are considered]

$$\begin{aligned}&= -\frac{MT}{OT} \cdot \frac{OT}{OP} - \frac{PR}{PT} \cdot \frac{PT}{OP} \\ &= -\sin TOM \cos POT - \cos RPT \sin POT \\ &= -\sin(180^\circ - A) \cos(B - 180^\circ) \\ &\quad - \cos(180^\circ - A) \sin(B - 180^\circ) \\ &= -\sin A (-\cos B) - (-\cos A)(-\sin B) \\ &= \sin A \cos B - \cos A \sin B.\end{aligned}$$

Similarly,

$$\begin{aligned}\cos(A - B) &= \frac{ON}{OP} \quad [\text{where } ON \text{ is taken with proper sign}] \\ &= -\frac{RT - OM}{OP} \quad [\text{where magnitudes only of } RT, OM \text{ etc. are considered}] \\ &= -\frac{RT}{PT} \cdot \frac{PT}{OP} + \frac{OM}{OT} \cdot \frac{OT}{OP} \\ &= -\sin RPT \sin POT + \cos TOM \cos POT \\ &= -\sin(180^\circ - A) \sin(B - 180^\circ) \\ &\quad + \cos(180^\circ - A) \cos(B - 180^\circ)\end{aligned}$$

$$= -\sin A (-\sin B) + (-\cos A)(-\cos B) \\ = \cos A \cos B + \sin A \sin B.$$

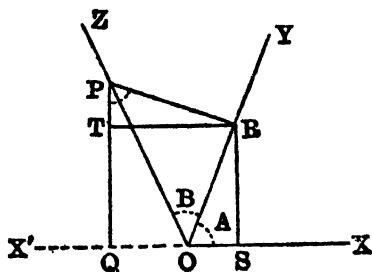
#### 4. A few particular cases of $\sin (A \pm B)$ , $\cos (A \pm B)$ .

*Case I.* In the case  $A$  and  $B$  are both acute and  $(A+B) > 90^\circ$ .

Construction is same as in Art. 33. Here,  $Q$ , the foot of the perpendicular will fall on  $XO$  produced.

$$\angle TPR = 90^\circ - \angle TRP = \angle TRO = \angle ROS = \angle A.$$

$$\begin{aligned}\sin(A+B) &= \sin XOP \\ &= \frac{PQ}{OP} = \frac{QT+TP}{OP} \\ &= \frac{RS+PT}{OP} = \frac{RS}{OP} + \frac{PT}{OP} \\ &= \frac{RS}{OR} \cdot \frac{OR}{OP} + \frac{PT}{PR} \cdot \frac{PR}{OP} \\ &= \sin A \cos B + \cos A \sin B.\end{aligned}$$



$$\begin{aligned}\cos (A+B) &= \cos XOP = -\frac{OQ}{OP} \quad [\text{Magnitude of } OQ \text{ being considered}] \\ &= -\frac{SQ - SO}{OP} = \frac{OS}{OP} - \frac{SQ}{OP} = \frac{OS}{OP} - \frac{TR}{OP}\end{aligned}$$

$$\begin{aligned}
 &= \frac{OS}{OR} \cdot \frac{OR}{OP} - \frac{TR}{PR} \cdot \frac{PR}{OP} \\
 &= \cos A \cos B - \sin TPR \sin B \\
 &= \cos A \cos B - \sin A \sin B.
 \end{aligned}$$

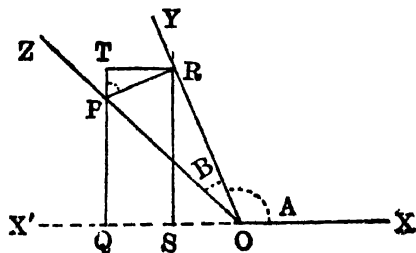
*Case II. In the case A is obtuse and B is acute and  $(A+B) < 180^\circ$ .*

Construction is same as in Art. 33.

Here,  $\angle TPR = 180^\circ - \angle RPQ = \angle ROQ = 180^\circ - A$ .

$$\therefore \sin TPR = \sin A; \cos TPR = -\cos A.$$

$$\begin{aligned}
 \sin(A+B) &= \sin XOP = \frac{PQ}{OP} = \frac{QT-PT}{OP} = \frac{RS-PT}{OP} \\
 &= \frac{RS}{OP} - \frac{PT}{OP} = \frac{RS}{OR} \cdot \frac{OR}{OP} - \frac{PT}{PR} \cdot \frac{PR}{OP} \\
 &= \sin A \cos B - \cos TPR \sin B. \\
 &= \sin A \cos B + \cos A \sin B.
 \end{aligned}$$



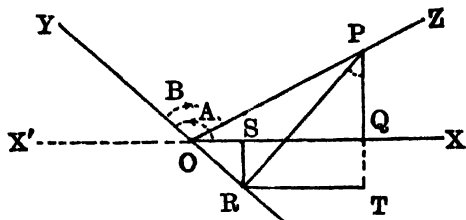
$$\begin{aligned}
 \cos(A+B) &= \cos XOP = -\frac{OQ}{OP} \quad [\text{Magnitude of } OQ \text{ being considered}] \\
 &= -\frac{OS+SQ}{OP} = -\frac{OS}{OP} - \frac{SQ}{OP} \\
 &= -\frac{OS}{OR} \cdot \frac{OR}{OP} - \frac{TR}{PR} \cdot \frac{PR}{OP} \\
 &= \cos A \cos B - \sin TPR \sin B \\
 &= \cos A \cos B - \sin A \sin B.
 \end{aligned}$$



*Case III.* In the case  $A$  and  $B$  are both obtuse and  $(A - B)$  is acute.

Construction is same as in Art. 34.

Here,  $\angle TPR = \angle ROS = 180^\circ - A$ .



$$\begin{aligned}
 \sin (A - B) &= \sin POQ \\
 &= \frac{PQ}{OP} = \frac{PT - RS}{OP} \\
 &= \frac{PT}{OP} - \frac{RS}{OP} = \frac{PT}{PR} \cdot \frac{PR}{OP} - \frac{RS}{OR} \cdot \frac{OR}{OP} \\
 &= \cos TPR \sin POR - \sin ROS \cos POR \\
 &= \cos (180^\circ - A) \sin (180^\circ - B) \\
 &\quad - \sin (180^\circ - A) \cos (180^\circ - B) \\
 &= -\cos A \sin B - \sin A (-\cos B) \\
 &= \sin A \cos B - \cos A \sin B.
 \end{aligned}$$

$$\begin{aligned}
 \cos (A - B) &= \cos POQ \\
 &= \frac{OQ}{OP} = \frac{OS + SQ}{OP} = \frac{OS + RT}{OP} = \frac{OS}{OP} + \frac{RT}{OP} \\
 &= \frac{OS}{OR} \cdot \frac{OR}{OP} + \frac{RT}{PR} \cdot \frac{PR}{OT} \\
 &= \cos ROS \cdot \cos POR + \sin TPR \cdot \sin POR \\
 &= \cos (180^\circ - A) \cos (180^\circ - B) \\
 &\quad + \sin (180^\circ - A) \sin (180^\circ - B)
 \end{aligned}$$

$$\begin{aligned}
 &= (-\cos A)(-\cos B) + \sin A \sin B \\
 &= \cos A \cos B + \sin A \sin B.
 \end{aligned}$$

**Note.** Other particular cases of the above four formulæ can easily be proved exactly in the same way by drawing the corresponding figures in each case and making the same construction as in Arts. 93 and 94 for  $(A+B)$  and  $(A-B)$  respectively.

**5. An alternative Method of proof for  $\sin(A \pm B)$ ,  $\cos(A \pm B)$ .** [ See Arts. 33, 34 ]

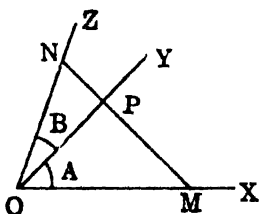


Fig. (i)

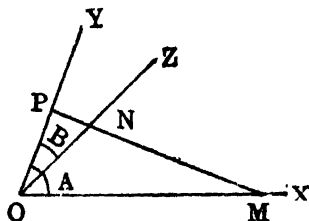


Fig. (ii)

Let  $\angle XOY = A$ ;  $\angle YOZ = B$ ; in Fig. (i),  $\angle XOZ = A+B$  ( $< 90^\circ$ ); in Fig. (ii),  $\angle XOZ = A-B$  ( $A > B$ ) [ $A$  and  $B$  being positive and acute].

Through any point  $P$  on  $OY$ , the common arm of two angles, draw a straight line  $MN$  perpendicular to  $OY$ , meeting  $OX$  in  $M$  and  $OZ$  in  $N$ .

Then,  $\triangle MON = \triangle MOP \pm \triangle NOP$ .

$$\therefore \frac{1}{2} OM \cdot ON \sin(A \pm B) = \frac{1}{2} OM \cdot OP \sin A \pm \frac{1}{2} ON \cdot OP \sin B$$

[ Art. 88(i) ]

$$\begin{aligned}
 \therefore \sin(A \pm B) &= \sin A \cdot \frac{OP}{ON} \pm \frac{OP}{OM} \sin B \\
 &= \sin A \cos B \pm \cos A \sin B.
 \end{aligned}$$

$$\begin{aligned}
 \cos(A \pm B) &= \cos MON = \frac{OM^2 + ON^2 - MN^2}{2OM \cdot ON} \quad [ \text{Art. 83} ] \\
 &= \frac{(OP^2 + PM^2) + (OP^2 + PN^2) - (MP \mp PN)^2}{2OM \cdot ON}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{OP^2 \mp MP \cdot PN}{OM \cdot ON} = \frac{OP}{OM} \cdot \frac{OP}{ON} \mp \frac{MP}{OM} \cdot \frac{PN}{ON} \\
 &= \cos A \cos B \mp \sin A \sin B.
 \end{aligned}$$

### 6. Geometrical proof of the expansion of $\tan (A+B)$ .

The figure and the construction are the same as in Art. 33.

$$\begin{aligned}
 \tan (A+B) &= \frac{PQ}{OQ} = \frac{RS+PT}{OS-TR} \\
 &= \frac{\frac{RS}{OS} + \frac{PT}{OS}}{1 - \frac{TR}{OS}} = \frac{\frac{RS}{OS} + \frac{PT}{OS}}{1 - \frac{TR}{TP} \cdot \frac{TP}{OS}}.
 \end{aligned}$$

Now,  $\frac{RS}{OS} = \tan A$  and  $\frac{TR}{TP} = \tan TPR = \tan A$ .

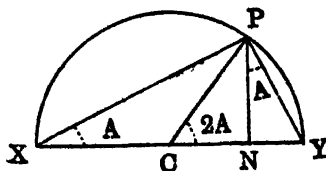
The triangles  $ROS$ ,  $TPR$  are similar.

$$\therefore \frac{TP}{OS} = \frac{PR}{OR} = \tan B.$$

$$\therefore \tan (A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

**Note.** Similarly the expansion of  $\tan (A-B)$  can be proved geometrically from the figure and construction of Art. 34.

### 7. Geometrical proof of the formulæ for $2A$ .



Let  $XPY$  be a semi-circle,  $XY$  the diameter and  $C$  the centre.

Draw  $PN$  perpendicular to  $XY$ .

Let  $\angle PXY = A$ ; then  $\angle PCY = 2A$ .

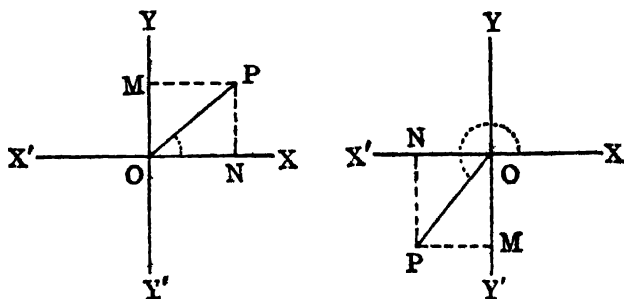
$$\angle NPY = 90^\circ - \angle PYN = \angle PXY = A.$$

$$\begin{aligned}\sin 2A &= \frac{PN}{CP} = \frac{2PN}{2CP} = \frac{2PN}{XY} = 2 \frac{PN}{XP} \cdot \frac{XP}{XY} \\ &= 2 \sin PYN \cdot \cos PXY = 2 \sin A \cos A.\end{aligned}$$

$$\begin{aligned}\cos 2A &= \frac{CN}{PC} = \frac{2CN}{2CP} = \frac{2CN}{XY} = \frac{CN + CN}{XY} \\ &= \frac{(XN - NC) + (CY - YN)}{XY} = \frac{XN - YN}{XY} \\ &= \frac{XN}{XP} \cdot \frac{XP}{XY} - \frac{YN}{PY} \cdot \frac{PY}{XY} \\ &= \cos A \cdot \cos A - \sin A \cdot \sin A \\ &= \cos^2 A - \sin^2 A.\end{aligned}$$

$$\begin{aligned}\tan 2A &= \frac{PN}{CN} = \frac{2PN}{2CN} = \frac{2PN}{XN - YN} \\ \frac{2 \frac{PN}{XN}}{1 - \frac{YN}{XN}} &= \frac{2 \frac{PN}{XN}}{1 - \frac{YN}{PN} \cdot \frac{PN}{XN}} \\ &= \frac{2 \tan A}{1 - \tan A \cdot \tan A} = \frac{2 \tan A}{1 - \tan^2 A}.\end{aligned}$$

**8. Trigonometrical ratios of generalized angle defined by projections.**



Let  $XOX'$  and  $YOY'$  be a pair of rectangular axes intersecting at the point  $O$  and let an angle  $\theta$ , of any magnitude (positive or negative) be generated by the revolution of  $OP$  from its initial position  $OX$  to its present position. Then the trigonometrical ratios of the generalized angle  $\theta$  are defined as follows :

$$\sin \theta = \frac{\text{projection of } OP \text{ on } y\text{-axis}}{OP}$$

$$\cos \theta = \frac{\text{projection of } OP \text{ on } x\text{-axis}}{OP}$$

$$\tan \theta = \frac{\text{projection of } OP \text{ on } y\text{-axis}}{\text{projection of } OP \text{ on } x\text{-axis}}$$

$$\operatorname{cosec} \theta = \frac{OP}{\text{projection of } OP \text{ on } y\text{-axis}}$$

$$\sec \theta = \frac{OP}{\text{projection of } OP \text{ on } x\text{-axis}}$$

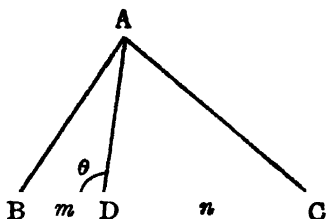
$$\cot \theta = \frac{\text{projection of } OP \text{ on } x\text{-axis}}{\text{projection of } OP \text{ on } y\text{-axis}}$$

In the above definitions, projection means algebraic projection ; that is, we should consider not only the magnitude but also the sign of the projection ; and with the usual convention the projection would be considered *positive* if they are along  $OX$  and  $OY$  and considered *negative* if they are along  $OX'$  and  $OY'$ . By convention,  $OP$  is *always considered positive*. From these definitions, the signs of the trigonometrical ratios can be easily determined according to the position of  $OP$  in one or other of the four quadrants. In the above figures, the position of  $OP$  in two quadrants only (1st and 3rd) are shown.

**Note 1.** From the above definitions, it is clear that if  $OX$  be a fixed line, and if  $l$  be the length of any straight line  $OP$  inclined at an angle  $\theta$  to  $OX$ , then the *projection of  $OP$  along  $OX$*  is  $l \cos \theta$  whatever be the magnitude of the angle  $\theta$ .

**Note 2.** The *Addition and Subtraction Theorem* for generalized angles can also be proved by the method of projection.

## 9. Two important trigonometrical relations.



If  $D$  be any point in the base  $BC$  of a triangle  $ABC$ , and if  $AD$  divides  $BC$  into two parts  $m$  and  $n$  ( $BD = m$ ,  $CD = n$ ) and the angle  $A$  into two parts  $\alpha$  and  $\beta$  ( $\angle BAD = \alpha$ ,  $\angle CAD = \beta$ ) and if the angle  $ADB$  be  $\theta$ , then

$$(i) \quad (m+n) \cot \theta = n \cot \beta - m \cot \alpha.$$

$$(ii) \quad (m+n) \cot \theta = m \cot C - n \cot B.$$

We have

$$\begin{aligned} \frac{m}{n} &= \frac{BD}{DC} = \frac{BD \cdot AD}{AD \cdot DC} = \frac{\sin BAD \cdot \sin ACD}{\sin ABD \cdot \sin DAC} \\ &= \frac{\sin \alpha \cdot \sin (\theta - \beta)}{\sin (\theta + \alpha) \cdot \sin \beta} \left[ \because \angle ABD = \pi - (\alpha + \theta), \right. \\ &\quad \left. \angle ACD = \theta - \beta. \right] \\ &= \frac{\sin \alpha (\sin \theta \cos \beta - \cos \theta \sin \beta)}{\sin \beta (\sin \theta \cos \alpha + \cos \theta \sin \alpha)} \end{aligned}$$

Dividing the numerator and the denominator by  $\sin \alpha \sin \beta \sin \theta$ , we have

$$\frac{m}{n} = \frac{\cot \beta - \cot \theta}{\cot \alpha + \cot \theta}.$$

$$\therefore (m+n) \cot \theta = n \cot \beta - m \cot \alpha.$$

Again,

$$\frac{m}{n} = \frac{\sin BAD \sin ACD}{\sin ABD \sin DAC}$$

$$\begin{aligned}
&= \frac{\sin(\theta+B)}{\sin B} \cdot \frac{\sin C}{\sin(\theta-C)} \left[ \because \angle BAD = \pi - (\theta+B), \right. \\
&\quad \left. \angle DAC = \theta - C \right] \\
&= \frac{\sin C (\sin \theta \cos B + \cos \theta \sin B)}{\sin B (\sin \theta \cos C - \cos \theta \sin C)}
\end{aligned}$$

Dividing the numerator and the denominator by  $\sin B \sin C \sin \theta$ , we have

$$\frac{m}{n} = \frac{\cot B + \cot \theta}{\cot C - \cot \theta}$$

$$\therefore (m+n) \cot \theta = m \cot C - n \cot B.$$

### 10. Note on Art. 90.

Let us denote the formulæ of Arts. 82, 83, 84 by (I), (II), (III). We have seen in Art. 90, that (II) can be deduced from (III). We shall now show how any one of these can be deduced from any other of the rest.

*To deduce (I) from (III).*

Substituting the value of  $b$  from the second relation of Art. 84 in the first,

$$\begin{aligned}
a &= (c \cos A + a \cos C) \cos C + c \cos B. \\
\therefore a(1 - \cos^2 C) &= c(\cos A \cos C + \cos B) \\
&= c\{\cos A \cos C - \cos(A+C)\} \\
&\quad [\because A+B+C=\pi]
\end{aligned}$$

$$\therefore a \sin^2 C = c \sin A \sin C. \quad \therefore a/\sin A = c/\sin C.$$

Similarly, substituting the value of  $c$  in the first relation, we get

$$a/\sin A = b/\sin B. \quad \text{Hence, etc.}$$

*To deduce (II) and (III) from (I).*

(i) Putting each of the ratios of Art. 82 equal to  $k$ , we get

$$\begin{aligned}
a &= k \cdot \sin A; \quad b = k \cdot \sin B; \quad c = k \cdot \sin C. \\
\therefore \frac{b^2 + c^2 - a^2}{2bc} &= \frac{k^2 (\sin^2 B + \sin^2 C - \sin^2 A)}{k^2 \cdot 2 \sin B \sin C} \\
&= \frac{\sin^2 B + \sin(C+A) \sin(C-A)}{2 \sin B \sin C}
\end{aligned}$$

$$\begin{aligned}
&= \frac{\sin B \{\sin B + \sin (C - A)\}}{2 \sin B \sin C} \\
&\quad [\because \sin (C + A) = \sin (\pi - B) = \sin B] \\
&= \frac{\sin B \{\sin (C + A) + \sin (C - A)\}}{2 \sin B \sin C} \\
&= \frac{2 \sin B \sin C \cos A}{2 \sin B \sin C} = \cos A.
\end{aligned}$$

$$\begin{aligned}
\text{(ii) } b \cos C + c \cos B &= k (\sin B \cos C + \sin C \cos B) \\
&= k \sin (B + C) = k \sin A \\
&= a. \quad [\because A + B + C = \pi]
\end{aligned}$$

To deduce (I) and (III) from (II).

$$\begin{aligned}
\text{(i) } \sin^2 A &= 1 - \cos^2 A \\
&= 1 - \left( \frac{b^2 + c^2 - a^2}{2bc} \right)^2 = \frac{4b^2c^2 - (b^2 + c^2 - a^2)^2}{4b^2c^2} \\
&= \frac{(2bc + b^2 + c^2 - a^2)(2bc - b^2 - c^2 + a^2)}{4b^2c^2} \\
&= \frac{(a + b + c)(b + c - a)(c + a - b)(a + b - c)}{4b^2c^2} \\
&= \frac{K}{4b^2c^2} \text{ say.}
\end{aligned}$$

$$\therefore \frac{\sin^2 A}{a^2} = \frac{K}{4a^2b^2c^2};$$

similarly,  $\frac{\sin^2 B}{b^2}$  and  $\frac{\sin^2 C}{c^2}$  each  $= \frac{K}{4a^2b^2c^2}$ .

$$\therefore \frac{\sin^2 A}{a^2} = \frac{\sin^2 B}{b^2} = \frac{\sin^2 C}{c^2}; \text{ hence, etc.}$$

(ii) Adding 2nd and 3rd relations of the formulæ of Art. 83, we get

$$b^2 + c^2 = b^2 + c^2 + 2a^2 - 2ca \cos B - 2ab \cos C.$$

Now, transposing and dividing by  $2a$ , we get

$$a = b \cos C + c \cos B; \text{ etc.}$$



### Miscellaneous Examples III

1. The angles of a triangle are as 4 : 5 : 6. Express them in circular measure.

2. The angles of a triangle are in A.P. and the greatest is double the least ; express the angles in degrees, and in radians.

3. The number of degrees in one of the acute angles of a right-angled triangle is three-tenths of the number of grades in the other ; determine the angles in degrees.

4. Compare the areas of two circles in which the circumference of one is equal to an arc of  $60^\circ$  of the other.

5. A railway train is travelling on a curve of half-a-mile radius at the rate of 20 miles an hour ; through what angle has it turned in 10 seconds ?

6. An arc of a circle whose radius is 7 inches, subtends an angle of  $15^\circ 39' 7''$  ; what angle will an arc of the same length subtend in a circle whose radius is 2 inches ?

Prove the following identities (*Ex. 7 to 22*) :—

$$7. \quad \sin^2 \theta \tan \theta + \cos^2 \theta \cot \theta + 2 \sin \theta \cos \theta = \tan \theta + \cot \theta.$$

$$8. \quad \sin^2 \theta (1 + \cot^2 \theta) + \cos^2 \theta (1 + \tan^2 \theta) = 2.$$

$$9. \quad (\tan \theta + \sec \theta)^2 = \frac{1 + \sin \theta}{1 - \sin \theta}. \quad [C. U. 1934]$$

$$10. \quad 2 (\sin^3 \theta + \cos^3 \theta) - 3 (\sin^4 \theta + \cos^4 \theta) + 1 = 0.$$

$$11. \quad \frac{\tan x - \cot y}{\tan y - \cot x} = \tan x \cot y.$$

$$12. \quad (\sin x \cos y + \cos x \sin y)^2 + (\cos x \cos y - \sin x \sin y)^2 = 1.$$

13.  $\sin^4 x + \cos^4 x = 1 - 2 \sin^2 x \cos^2 x.$
14.  $\sin^8 x - \cos^8 x = (\sin^2 x - \cos^2 x)(1 - 2 \sin^2 x \cos^2 x).$
15.  $\sqrt{\frac{1 - \sin \theta}{1 + \sin \theta}} = \sec \theta - \tan \theta.$
16.  $(1 + \sec \theta + \tan \theta)(1 - \sec \theta + \tan \theta) = 2 \tan \theta.$
17.  $\frac{\cos \theta}{1 - \tan \theta} + \frac{\sin \theta}{1 - \cot \theta} = \sin \theta + \cos \theta.$
18.  $(\sin x + \cos x)^2 + (\sin x - \cos x)^2 = 2.$
19.  $\cot^2 x \cdot \frac{\sec x - 1}{1 + \sin x} + \sec^2 x \cdot \frac{\sin x - 1}{1 + \sec x} = 0.$
20.  $(\sin x + \cos x)(\tan x + \cot x) = \sec x + \operatorname{cosec} x.$
21.  $(\sin \theta + \sec \theta)^2 + (\cos \theta + \operatorname{cosec} \theta)^2 = (1 + \sec \theta \operatorname{cosec} \theta)^2.$
22.  $\frac{1 - \sin \theta \cos \theta}{\cos \theta (\sec \theta - \operatorname{cosec} \theta)} \cdot \frac{\sin^2 \theta - \cos^2 \theta}{\sin^3 \theta + \cos^3 \theta} = \sin \theta.$
23. If  $a \cos^2 x + b \sin^2 x = c$ , show that  $\tan x = \pm \sqrt{\frac{c-a}{b-c}}.$
24. If  $\operatorname{cosec} A + \operatorname{cosec} B + \operatorname{cosec} C = 0$ , show that  
 $(\Sigma \sin A)^2 = \Sigma \sin^2 A.$
25. If  $x = a \cos \theta + b \sin \theta$  and  $y = a \sin \theta - b \cos \theta$ ,  
show that  $x^2 + y^2 = a^2 + b^2.$
26. Express  $\frac{\sin x}{\cos^3 x} + \frac{\cos x}{\sin^3 x}$  in terms of  $t$ ,  
where  $t$  stands for  $\tan x.$
27. If  $\sin A = \frac{1}{2}$  and  $\tan B = \sqrt{3}$ , find the value of  
 $\sin A \cos B + \cos A \sin B.$
28. If  $\cos \theta = \frac{4}{5}$ , find the value of  $\frac{1 - \tan^2 \theta}{1 + \tan^2 \theta}.$
29. If  $5 \tan \theta = 4$ , find the value of  
 $\frac{5 \sin \theta - 3 \cos \theta}{\sin \theta + 2 \cos \theta}.$

30. If  $\frac{\sin x}{\sin y} = \sqrt{2}$ ,  $\frac{\tan x}{\tan y} = \sqrt{3}$ ,

find  $x$  and  $y$  (given that  $x$  and  $y$  are acute angles).

31. Which of the statements is possible and which impossible,  $x$ ,  $y$  and  $z$  being unequal real quantities ?

(i)  $\operatorname{cosec} \theta = \frac{x^2 + y^2}{2xy}$ . (ii)  $\sec \theta = \frac{x^2 - y^2}{x^2 + y^2}$ .

(iii)  $\sin \theta = \frac{x^2 + y^2 + z^2}{yz + zx + xy}$ . (iv)  $\tan \theta = \frac{2xy}{x^2 + y^2}$ .

32. Eliminate  $\theta$  from the equations :

(i)  $\sin \theta + \cos \theta = a$ ,  $\sin \theta - \cos \theta = b$ .

(ii)  $\frac{x}{a} \cos \theta + \frac{y}{b} \sin \theta = 1$ ,  $\frac{x}{a} \sin \theta - \frac{y}{b} \cos \theta = 1$ .

(iii)  $x = a \cos^3 \theta$ ,  $y = b \sin^3 \theta$ .

33. If  $k \tan \theta = \tan k\theta$ , prove that

$$\frac{\sin^2 k\theta}{\sin^2 \theta} = \frac{k^2}{1 + (k^2 - 1) \sin^2 \theta}.$$

34. If  $\sec x \sec y + \tan x \tan y = \sec z$ ,

$$\text{then, } \sec x \tan y + \tan x \sec y = \pm \tan z.$$

35. Show that  $\left( \frac{1 + \cot 60^\circ}{1 - \cot 60^\circ} \right)^2 = \frac{1 + \cos 30^\circ}{1 - \cos 30^\circ}$ .

36. If  $\tan x = \frac{\sin \theta - \cos \theta}{\sin \theta + \cos \theta}$ , prove that

$$\sin x = \frac{1}{\sqrt{2}} (\sin \theta - \cos \theta).$$

37. Show that the product of  $\sin x (1 + \sin x) + \cos x (1 + \cos x)$  and  $\sin x (1 - \sin x) + \cos x (1 - \cos x)$  is equal to  $2 \sin x \cos x$ .

38. Find the height of a chimney when it is found that on walking towards it 250 feet, in a horizontal line through its base, the angular elevation changes from  $45^\circ$  to  $75^\circ$ .

39. The length of a kite string is 250 yards, and the angle of elevation of the kite is  $30^\circ$ . Find the height of the kite.

40. The angle of elevation of the top of a temple at a distance 300 feet is  $30^\circ$ ; find its height.

41. Find the angle of elevation of the sun when the shadow of a pole 60 feet high, is  $20\sqrt{3}$  yards long.

42. The angles of elevation of a tower at two places due east of it and 200 feet apart are  $45^\circ$  and  $30^\circ$ ; find the height of the tower.

43. An aeroplane leaves the earth at the point  $X$  and rises at a uniform speed. At the end of 15 seconds, an observer stationed at a distance of 660 feet from  $X$ , find the angular elevation of the aeroplane to be  $45^\circ$ ; at what rate in miles per hour is the aeroplane rising?

44. A ladder 45 feet long just reaches the top of a wall. The ladder makes an angle of  $60^\circ$  with the wall. Find the height of the wall and the distance of the foot of the ladder from the wall.

45. If  $\cos A = \frac{4}{5}$ ,  $\cos B = \frac{3}{5}$ , find the values of  $\sin(A+B)$  and  $\cos(A-B)$ .

46. If  $\tan A = \frac{5}{12}$  and  $\tan B = \frac{9}{16}$ , find the values of  $\sin(A-B)$  and  $\cos(A-B)$ .

47. If  $\tan A = \frac{m+n}{m-n}$ , and  $\tan B = \frac{m-n}{m+n}$ , find  $\tan(A-B)$ .

48. If  $\tan(x+y) = \frac{3}{4}$  and  $\tan x = \frac{5}{6}$ , find  $\tan y$ .

49. If  $\cos \theta = \frac{3}{5}$ , find  $\sin 2\theta$ ,  $\tan 2\theta$ ,  $\cos \frac{\theta}{2}$ .

50. If  $\cos x = \frac{4}{5}$ ,  $\cos y = \frac{3}{5}$  ( $x$  and  $y$  being positive acute angles), find the value of  $\cos \frac{1}{2}(x-y)$ .

51. If  $\sin A = \frac{1}{\sqrt{2}}$ ,  $\sin B = \frac{1}{\sqrt{3}}$ , find the value of

$$\tan \frac{1}{2} (A+B) \cot \frac{1}{2} (A-B).$$

52. If  $\sec x = \frac{17}{8}$ ,  $\operatorname{cosec} y = \frac{5}{2}$ , find  $\sec (x+y)$ .

53. Prove that

$$\frac{2 \cos 8\theta + 1}{2 \cos \theta + 1} = (2 \cos \theta - 1)(2 \cos 2\theta - 1)(2 \cos 4\theta - 1).$$

54. Show that  $a \cos \theta + b \sin \theta = \sqrt{a^2 + b^2} \cos (\theta - \alpha)$ , if  $\tan \alpha = b/a$ .

55. If  $\sin^4 x + \cos^4 x = 1$ , prove that  $x$  is zero or a multiple of  $\frac{1}{2}\pi$ .

56. If  $\sqrt{2} \cos A = \cos B + \cos^3 B$ , and  $\sqrt{2} \sin A = \sin B - \sin^3 B$ , then  $\sin (A-B) = \pm \frac{1}{2}$ .

57. Prove that  $\cos^2 (\alpha - \beta) - \sin^2 (\alpha + \beta) = \cos 2\alpha \cos 2\beta$ .

58. Show that  $\sin 18^\circ + \cos 18^\circ = \sqrt{2} \cos 27^\circ$ .

59. Show that whatever be the value of  $\theta$ ,

$\sin^2(\theta + \alpha) + \sin^2(\theta + \beta) - 2 \cos (\alpha - \beta) \sin (\theta + \alpha) \sin (\theta + \beta)$  is independent of  $\theta$ .

60. Show that

$$\begin{aligned} \text{(i)} \quad & \frac{\sin \alpha}{\sin (\alpha - \beta) \sin (\alpha - \gamma)} + \frac{\sin \beta}{\sin (\beta - \gamma) \sin (\beta - \alpha)} \\ & + \frac{\sin \gamma}{\sin (\gamma - \alpha) \sin (\gamma - \beta)} = 0. \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad & \tan (\beta + \gamma - 2\alpha) + \tan (\gamma + \alpha - 2\beta) + \tan (\alpha + \beta - 2\gamma) \\ & = \tan (\beta + \gamma - 2\alpha) \tan (\gamma + \alpha - 2\beta) \tan (\alpha + \beta - 2\gamma). \end{aligned}$$

61. If  $\tan \frac{1}{2}\theta = \tan^2 \frac{1}{2}\phi$  and  $\tan \phi = 2 \tan \alpha$ , show that

$$\theta + \phi = 2\alpha.$$

62. (i) If  $\tan^2 x = 2 \tan^2 y + 1$ , then  $\cos 2x + \sin^2 y = 0$ .

(ii) If  $\cos A = \tan B$ ,  $\cos B = \tan C$ ,  $\cos C = \tan A$ ,  
prove that  $\sin A = \sin B = \sin C$ .

63. Show that  $\tan 20^\circ \tan 40^\circ \tan 80^\circ = \tan 60^\circ$ .

64. If  $\alpha + \beta + \gamma = 0$ , prove that

$$\cos \alpha + \cos \beta + \cos \gamma = 4 \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2} - 1.$$

65. If in any triangle,  $\tan \phi = \frac{a-b}{a+b} \cot \frac{1}{2}C$ , prove that

$$c = (a+b) \sin \frac{1}{2}C \sec \phi.$$

66. If  $\cos \theta = \frac{a \cos \phi - b}{a - b \cos \phi}$ , then  $\frac{\tan \frac{\theta}{2}}{\sqrt{a+b}} = \frac{\tan \frac{\phi}{2}}{\sqrt{a-b}}$ .

67. If  $\alpha + \beta + \gamma = \frac{1}{2}\pi$ , prove that

$$\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma + 2 \sin \alpha \sin \beta \sin \gamma = 1.$$

68. If  $\cos^2 A + \cos^2 B + \cos^2 C + 2 \cos A \cos B \cos C = 1$ ,  
show that one of the quantities  $A \pm B \pm C$  is an odd multiple  
of  $\pi$ .

69. Show that  $\sec x = \frac{2}{\sqrt{2+\sqrt{2+2\cos 4x}}}$ .

70. If  $a \sin (\theta + \alpha) = b \sin (\theta + \beta)$ , prove that

$$\cot \theta = \frac{a \cos \alpha - b \cos \beta}{b \sin \beta - a \sin \alpha}.$$

71. If  $\tan \beta = \frac{n \sin \alpha \cos \alpha}{1 - n \sin^2 \alpha}$ , show that

$$\tan (\alpha - \beta) = (1 - n) \tan \alpha.$$

In any triangle, prove that (Ex. 72 to 77) :

$$72. \quad \frac{\cos A}{c \cos B + b \cos C} + \frac{\cos B}{a \cos C + c \cos A} + \frac{\cos C}{b \cos A + a \cos B} = \frac{a^2 + b^2 + c^2}{2abc}.$$

$$73. \quad \frac{\tan \frac{A}{2}}{(a-b)(a-c)} + \frac{\tan \frac{B}{2}}{(b-c)(b-a)} + \frac{\tan \frac{C}{2}}{(c-a)(c-b)} = \frac{1}{\Delta}.$$

$$74. \quad \sin 3A \sin (B-C) + \sin 3B \sin (C-A) + \sin 3C \sin (A-B) = 0.$$

$$75. \quad \cot B + \frac{\cos C}{\sin B \cos A} = \cot C + \frac{\cos B}{\sin C \cos A}.$$

$$76. \quad c = (a-b) \sec \theta, \text{ where } \tan \theta = \frac{2\sqrt{ab}}{a-b} \sin \frac{C}{2}.$$

$$77. \quad a(\cos B \cos C + \cos A) = b(\cos C \cos A + \cos B) = c(\cos A \cos B + \cos C).$$

78. If in a triangle,  $c(a+b) \cos \frac{B}{2} = b(a+c) \cos \frac{C}{2}$ , show that the triangle is isosceles.

79. If in a triangle,  $a, b, c$  be in A.P. and the greatest angle exceeds the least by  $90^\circ$ , prove that

$$a : b : c = \sqrt{7} - 1 : \sqrt{7} : \sqrt{7} + 1.$$

80. In the solution of triangles there can be no ambiguity except when an angle is determined by the sine or cosecant, and in no case whatever, when the triangle has one right angle; prove this. [Cambridge]

81. If  $\sin(\pi \cos \theta) = \cos(\pi \sin \theta)$ , prove that

$$\cos\left(\theta \pm \frac{\pi}{4}\right) = \frac{1}{2\sqrt{2}}.$$

82. If  $\sin(\pi \cot \theta) = \cos(\pi \tan \theta)$ , prove that either  $\operatorname{cosec} 2\theta$  or  $\cot 2\theta$  is equal to  $n + \frac{1}{2}$ ,  $n$  being an integer.

83. If  $\alpha$  and  $\beta$  be the different values of  $\theta$  which satisfy the equation  $a \cos \theta + b \sin \theta = c$ , prove that

$$\sin(\alpha + \beta) = \frac{2ab}{a^2 + b^2}.$$

84. Find all the values of  $\theta$  which satisfy the equation  
 $\sin \theta + \sin 2\theta + \sin 3\theta = 1 + \cos \theta + \cos 2\theta.$

85. Prove that in any triangle,

$$\frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab} = \frac{1}{2Rr}.$$

86. If  $r : R : r_1 = 2 : 5 : 12$ , show that the triangle is right-angled.

87. If the angle of elevation of a cloud observed from a point at a height  $h$  above the surface of a lake be  $\phi$  and the angle of depression of its image in the lake be  $\theta$ , prove that the height of the cloud above the lake is  $h \frac{\sin(\theta + \phi)}{\sin(\theta - \phi)}$ , assuming that the image is vertically as much below the surface as the cloud is above it.

[ *B. H. U. I. 1931 ; A. U. 1942* ]

88. The elevation of a tower due north of a station at  $A$  is  $\alpha$  and at a station  $B$  due west of  $A$  is  $\beta$ . Prove that its altitude is  $\frac{AB \sin \alpha \sin \beta}{\sqrt{\sin^2 \alpha - \sin^2 \beta}}$ . [ *B. H. U. I. 1934* ]

89. A man walks along a straight road and observes that the greatest angle subtended by two objects is  $\alpha$ ; from the point where this greatest angle is subtended, he walks a distance  $c$  along the road and finds that the two objects are now in a straight line which makes an angle  $\beta$  with the road. Prove that the distance between the objects is  $c \sin \alpha \sin \beta \sec \frac{\alpha + \beta}{2} \sec \frac{\alpha - \beta}{2}$ . [ *B. H. U. I. 1936* ]

90. On the bank of a river is a column 200 ft. high supporting a statue 30 ft. high. To an observer on the opposite bank with his eye on the level of the ground the statue subtends an angle equal to that subtended by a man 6 ft. high standing at the base of the column; determine the breadth of the river. [ *B. H. U. I. 1941* ]



91. If  $O$  be a point inside the triangle  $ABC$  such that the angles  $AOB$ ,  $BOC$ ,  $COA$  are each equal to  $120^\circ$  and if  $OA = x$ ,  $OB = y$  and  $OC = z$ , show that

$$a^2(y-z) + b^2(z-x) + c^2(x-y) = 0.$$

92. If  $\tan \theta = \frac{\tan x + \tan y}{1 + \tan x \tan y}$ , prove that

$$\sin 2\theta = \frac{\sin 2x + \sin 2y}{1 + \sin 2x \sin 2y}.$$

93. Show that

$$\frac{\sin \theta - \sin 3\theta}{\cos \theta + \cos 3\theta} + \frac{\sin \theta + \sin 3\theta}{\cos \theta - \cos 3\theta} = 2 \cot 2\theta.$$

94. Prove that in the triangle  $ABC$

$$\frac{a^2(b^2 + c^2 - a^2)}{\sin 2A} = \frac{b^2(c^2 + a^2 - b^2)}{\sin 2B} = \frac{c^2(a^2 + b^2 - c^2)}{\sin 2C}.$$

95. Show that

$$\cos^4 \frac{\pi}{8} + \cos^4 \frac{3\pi}{8} + \cos^4 \frac{5\pi}{8} + \cos^4 \frac{7\pi}{8} = \frac{3}{2}.$$

96. If  $x + y = 3 - \cos 4\theta$ , and  $x - y = 4 \sin 2\theta$ ,  
show that  $\sqrt{x} + \sqrt{y} = 2$ .

97. Show that

$$\cot B \cot C + \cot C \cot A + \cot A \cot B = 1$$

where  $A = 2\beta + \gamma - 3\alpha$ ,  $B = 2\gamma + \alpha - 3\beta$ ,  $C = 2\alpha + \beta - 3\gamma$ .

98. Prove that in the triangle  $ABC$ ,

$$BC^2 \cos 2B + CA^2 \cos 2A + 2BC \cdot CA \cos (A - B) = AB^2.$$


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# ANSWERS

## Examples I. [ Pages 11-14 ]

1. (i) first quadrant. (ii) third quadrant.  
(iii) second quadrant. (iv) fourth quadrant.
2. (i)  $61^\circ 34' 44'' \cdot 4$ . (ii)  $175^\circ 49' 1'' \cdot 776$ .
3. (i)  $253775\pi$ . (ii)  $3\frac{2}{3}\pi$ .
4.  $82^\circ 30'$ ;  $91^\circ 66' 6'' \cdot 6$ ;  $\frac{1}{4}\pi$ . 5.  $\alpha : \beta = 5\pi : 24$ .
6.  $\frac{1}{2} \left( 1 - \frac{\pi}{180} \right)$ . 7.  $6^\circ$  and  $9^\circ$ . 8.  $\frac{1}{90} \left( D + \frac{M}{60} \right) - \frac{1}{100} \left( G + \frac{n}{100} \right)$ .
9.  $\frac{180}{1155}$  nearly. 10.  $20^\circ$  and  $30^\circ$ . 12.  $20^\circ, 40^\circ, 80^\circ$ .
13.  $27^\circ, 9^\circ, 18^\circ$ . 14. (i) At  $28\frac{1}{4}$  min. and 48 min. past 7.  
(ii) At 7-10. 15.  $20^\circ, 60^\circ, 100^\circ$ . 16.  $\frac{\pi}{7}, \frac{2\pi}{7}, \frac{4\pi}{7}; \frac{\pi}{21}, \frac{4\pi}{21}, \frac{16\pi}{21}$ .
17.  $45^\circ, 60^\circ, 120^\circ, 185^\circ$ . 18. 9.
19.  $mx$  and  $nx$  where  $x = \frac{2(10pn - 9qn)}{mn(10p - 9q)}$ . 20.  $\frac{2}{3}$ .
21. 3 and 6. 22.  $51 \cdot 41$  miles per hour (nearly).
23.  $66444$  miles per hour (nearly);  $431445$  miles (nearly).
24.  $76 \cdot 8$  ft. (nearly). 25.  $3959$  miles (nearly). 26.  $33$  ft.
27.  $360$  yds.

## Examples II. [ Pages 24-26 ]

25.  $(\sin \theta - \cos \theta)^2$ . 26.  $\frac{1}{\tan^2 \theta} - \tan^2 \theta$ . 31.  $\frac{a^2 - b^2}{a^2 + b^2}$ .
33.  $\pm \frac{\sqrt{\sec^2 a - 1}}{\sec a}$ ;  $\pm \frac{\sqrt{1 + \cot^2 \theta}}{\cot \theta}$ . 34.  $\frac{1}{2}$ . 36.  $\frac{1}{2}$ . 37. 1 or  $\frac{1}{2}$ .
39.  $\frac{a^2 - b^2}{2ab}$ ;  $\frac{a^2 + b^2}{a^2 - b^2}$ . 43. (i)  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ . (ii)  $xy = c^2$ .
- (iii)  $(bc' - b'c)^2 + (ca' - c'a)^2 = (ab' - a'b)^2$ .
- (iv)  $(a'b - b'c)(ab' - bc') = (aa' - cc')^2$ .

**Examples III. [ Pages 35-36 ]**

7.  $\frac{\sqrt{3}}{2}$ .      8. (i)  $60^\circ$ .      (ii)  $45^\circ$ .      (iii)  $30^\circ$  (There is another angle which is not one of the standard angles).  
 (iv)  $45^\circ$ .      (v)  $30^\circ$ .      (vi)  $30^\circ$ .      (vii)  $30^\circ$ .  
 9.  $\theta = 52\frac{1}{2}^\circ$ ,  $\phi = 7\frac{1}{2}^\circ$ .      10.  $\alpha = 50^\circ$ ,  $\beta = 10^\circ$ .  
 11.  $A = 22\frac{1}{2}^\circ$ ,  $B = 67\frac{1}{2}^\circ$ ,  $C = 45^\circ$ .      12. (i)  $-\frac{1}{3}$ .      (ii) 1.

**Examples IV. [ Pages 49-51 ]**

1.  $\frac{1}{2}$ ;  $-\frac{1}{\sqrt{3}}$ ;  $\frac{2}{\sqrt{3}}$ ; -1.      2.  $-\frac{1}{\sqrt{2}}$ ;  $-\frac{2}{\sqrt{3}}$ ;  $-\frac{1}{\sqrt{3}}$ ;  $\frac{\sqrt{3}}{2}$ .      3. 0.  
 4.  $\frac{\sqrt{3}}{2}$ .      5. (i) 1.      (ii)  $\pm 2$ ,  $\pm \frac{2}{\sqrt{3}}$ .      10.  $\tan^2 \theta$ ; 1.      12. (i) 2.  
 (ii) 1.      (iii)  $\sin x$  or 0 according as  $n$  is odd or even.      13.  $\frac{5}{3}$ .  
 14.  $\frac{\sqrt{40}}{3}$ .      15. (i)  $\cot 26^\circ$ .      (ii)  $\cos 25^\circ$ .      (iii)  $\operatorname{cosec} 39^\circ$ .      (iv)  $\cos \frac{\pi}{9}$ .  
 16. (i)  $300^\circ$ .      (ii)  $480^\circ$ .      17. (i)  $60^\circ$ .      (ii)  $120^\circ$ ,  $240^\circ$ .  
 (iii)  $30^\circ$ ,  $150^\circ$ ,  $210^\circ$ ,  $330^\circ$ .      (iv)  $30^\circ$ ,  $150^\circ$ .      (v)  $30^\circ$ ,  $135^\circ$ ,  $150^\circ$ ,  $315^\circ$ .

**Examples V. [ Pages 56-59 ]**

1.  $100\sqrt{3}$  ft.      2.  $2'89\ldots$  miles;  $2\frac{1}{2}$  miles.      3.  $20\sqrt{3}$  ft.; 120 ft.  
 4.  $20\sqrt{3}$  ft.; 20 ft.      5.  $30\sqrt{2}$  ft.      6.  $400(\sqrt{3}+1)$  yds.  
 7.  $40\sqrt{3}$  ft.      8.  $\frac{1}{2}(3 \pm \sqrt{3})$  miles.      9. 22.3 miles nearly.  
 10.  $94'64$  ft. nearly.      11.  $47'32$  ft. nearly.      12. 60 miles per hour.  
 13.  $50\sqrt{6}$  ft.      14.  $40\sqrt{6}$  ft.;  $40\sqrt{2}(\sqrt{7}+1)$  ft.  
 15.  $\frac{1}{2}(\sqrt{3}+1)$  miles.      16.  $5\sqrt{13}$  miles.  
 17.  $241'6\ldots$  ft.;  $91'6\ldots$  ft.      18.  $5'25\ldots$  miles per hour.  
 19.  $367'38$  ft.      20.  $\frac{1}{2}\sqrt{6}(\sqrt{5}+1)$ .  
 22. 2 miles.      23.  $13'66$  ft.

**Examples VI. [ Pages 68-70 ]**

21.  $\sin A \cos B \cos C - \sin B \cos C \cos A + \sin C \cos A \cos B$   
 $+ \sin A \sin B \sin C$ ;  

$$\frac{\tan A - \tan B - \tan C - \tan A \tan B \tan C}{1 + \tan A \tan B + \tan C \tan A - \tan B \tan C}$$
  
 22. 
$$\frac{\cot A \cot B \cot C - \cot A - \cot B - \cot C}{\cot B \cot C + \cot C \cot A + \cot A \cot B - 1}$$

**Examples VIII. [ Pages 79-81 ]**

27. (i)  $a$ .

## Examples IX. [Pages 86-87]

$$16. \frac{b^2 - a^2}{b^2 + a^2}. \quad 17. (i) 2 \sin \frac{1}{2}A \sqrt{1 + \sin A} + \sqrt{1 - \sin A}.$$

$$(ii) \text{ No ; } 2 \sin \frac{1}{2}\theta = \sqrt{1 + \sin \theta} + \sqrt{1 - \sin \theta}.$$

## Examples XI. [Pages 110-111]

$$1. n\pi \pm \frac{\pi}{4}, \text{ i.e., } (2k+1)\frac{\pi}{4}. \quad 2. (i) n\pi \pm \frac{\pi}{4}. \quad (ii) n\pi \pm \frac{\pi}{3}.$$

$$3. 2n\pi \pm \frac{\pi}{3}, (2k+1)\pi. \quad 4. \frac{n\pi}{2} + (-1)^n \frac{\pi}{12}.$$

$$5. n\pi + \frac{\pi}{4}, \text{ or, } n\pi + (-1)^n \frac{\pi}{6}. \quad 6. \frac{n\pi}{3}, \text{ or, } n\pi \pm \frac{\pi}{6}.$$

$$7. \frac{n\pi}{n+(-1)^n}. \quad 8. (2n+1)\frac{\pi}{2}, \text{ or, } (2n+1)\frac{\pi}{4}, \text{ or, } (2n+1)\frac{\pi}{8}.$$

$$9. n\pi - \frac{\pi}{4}, \text{ or, } \frac{n\pi}{2} + (-1)^n \frac{\alpha}{2}, \text{ where } \sin \alpha = \frac{\sqrt{5}-1}{2}. \quad 10. \frac{1}{4}\pi.$$

$$11. n\pi + \frac{\pi}{4}. \quad 12. (4n+1)\frac{\pi}{8}. \quad 13. 2n\pi + \frac{5\pi}{12}, \text{ or, } 2n\pi - \frac{\pi}{12}.$$

$$14. (2n+1)\frac{\pi}{4}, \text{ or, } n\pi \pm \frac{\pi}{6}. \quad 15. 2n\pi + \frac{\pi}{2}, \text{ or, } 2n\pi - \beta, \text{ where } \beta \text{ is a positive acute angle whose sine is } \frac{1}{2}. \quad 16. \frac{1}{2}n\pi. \quad 17. n\pi \pm \frac{1}{2}\pi.$$

$$18. (4n+1)\frac{\pi}{12}, (n \neq 3m+2). \quad 19. 2n\pi + \frac{1}{3}\pi, \text{ or, } 2n\pi + \frac{1}{3}\pi.$$

$$20. -\frac{1}{2}\pi, -\frac{1}{6}\pi, \frac{1}{2}\pi, \frac{1}{6}\pi. \quad 21. \frac{1}{2}(n\pi + \alpha), \text{ where } \tan \alpha = 2.$$

$$22. 2n\pi. \quad 23. 2n\pi, \frac{1}{6}(4n+1)\pi. \quad 24. 90^\circ, 450^\circ, 810^\circ.$$

$$25. \frac{1}{2}\pi, \frac{1}{2}\pi. \quad 27. (i) \frac{1}{2}n\pi + \frac{1}{4}\pi; 2n\pi \pm \frac{3}{2}\pi.$$

$$(ii) 0, \pm \frac{\pi}{12}, \pm \frac{\pi}{6}, \pm \frac{\pi}{4}. \quad (iii) \frac{n\pi}{3}; n\pi \pm \tan^{-1} \frac{1}{\sqrt{2}}. \quad (iv) 2n\pi - \alpha, \frac{4n-1}{2}\pi + \alpha.$$

$$(v) 2n\pi, \text{ or, } 2n\pi - \frac{1}{2}\pi. \quad (vi) (2n+1)\frac{\pi}{2}, \frac{4n+1}{14}\pi, \frac{4n-1}{6}\pi.$$

$$(vii) n\pi + \frac{\alpha}{2}; (2n+1)\frac{\pi}{6} - \frac{\alpha}{6}. \quad 28. n\pi + (-1)^n 21^\circ 48' - 68^\circ 12'.$$

$$29. (i) \alpha = \beta = \frac{1}{2}\pi, \text{ or, } \alpha = \frac{3}{2}\pi, \beta = -\frac{1}{2}\pi.$$

$$(ii) \alpha = \frac{1}{2}\pi, \beta = \frac{1}{2}\pi; \text{ or, } \alpha = \frac{1}{2}\pi, \beta = \frac{3}{2}\pi;$$

$$\text{or, } \alpha = \frac{3}{2}\pi, \beta = \frac{1}{2}\pi; \text{ or, } \alpha = \frac{1}{2}\pi, \beta = -\frac{1}{2}\pi.$$

## Examples XII. [Pages 119-121]

$$22. (i) 1. \quad (ii) 0. \quad (iii) \frac{x+y}{1-xy}. \quad 23. y = \frac{4x(1-x^2)}{1-6x^2+x^4}.$$

$$24. (x-y)(1+yz) = (y-z)(1+xy). \quad 25. (i) \frac{1}{2}, \text{ or, } -8. \quad (ii) \frac{a-b}{1+ab}.$$

- (iii)  $\pm \sqrt[5]{5}$ . (iv)  $\pm \frac{1}{\sqrt{3}}$ . (v)  $\frac{1}{2}$ , or,  $-\frac{1}{2}$ . (vi)  $\pm \frac{1}{\sqrt{2}} \sqrt{21}$   
 (vii) 0, or,  $\frac{1}{2}$ . (viii) 0,  $\pm \frac{1}{2}$ . (ix)  $2 - \sqrt{3}$ . (x)  $\frac{6 + \sqrt{6}}{8}$ .

### Miscellaneous Examples I. [ Pages 122-123 ]

2.  $\pm \sqrt{\frac{b^2 - c^2}{a^2 - c^2}}$ . 19.  $a^2 + b^2 = 2(1 + c)$ .

### Examples XIII(a). [ Pages 135-137 ]

1. (i) 6. (ii) -3. 2. -2. 5.  $\frac{n}{n-1}$ . 9. (i) 1. (ii)  $1\frac{1}{2}$ .  
 10.  $\bar{1}1173942, \cdot 3861209$ . 13.  $2\cdot 425805$ . 14.  $\cdot 41369$ .  
 15. (i)  $\bar{1}8969092$ . (ii)  $\cdot 898665$ . 16.  $39\cdot 879$ .  
 17. (i) 18. (ii) 6. (iii) 25. 18. (i) 24. (ii) 4. (iii) 79.  
 19. (i)  $\frac{\log 2}{\log 3}$ , i.e.,  $\cdot 63\ldots$  (ii)  $4 + \frac{\log 7}{\log 8}$ , i.e.,  $5\cdot 77\ldots$   
 (iii)  $\frac{2 \log 7 - 3 \log 3}{6 \log 5 - \log 7 - 2 \log 3}$ , i.e.,  $\cdot 108\ldots$   
 (iv)  $x = \frac{\log 3}{\log 3 - \log 2} = 2\cdot 71$  nearly,  $y = \frac{\log 2}{\log 3 - \log 2} = 1\cdot 71$  nearly.  
 (v)  $\frac{2b(2a-b)}{5ab+3ac-2b^2-bc}$  and  $\frac{2ab}{5ab+3ac-2b^2-bc}$ ,  
 where  $a = \log 2$ ,  $b = \log 3$ ,  $c = \log 7$ .  
 20. (i)  $\log x = \frac{a+3b}{5}$ ;  $\log y = \frac{a-2b}{5}$ .

### Examples XIII(b). [ Pages 142-144 ]

1.  $3\cdot 2766077$ . 2.  $1\cdot 3686646$ . 3.  $3\cdot 76018$ . 4.  $\cdot 7400827$ .  
 5.  $\cdot 8455104$ ;  $82^\circ 17' 21''$ . 6.  $\cdot 7928868$ .  
 7.  $9\cdot 8440554$ ,  $10\cdot 1559446$ . 8.  $36^\circ 24' 36''$ .  
 9.  $58^\circ 13' 55''$ . 10.  $9\cdot 6198509$ ;  $22^\circ 36' 28''$ .  
 12.  $10\cdot 0957689$ . 13.  $9\cdot 9147384$ . 14.  $9\cdot 8718486$ .  
 16.  $\theta = 50^\circ 7' 48''$  nearly. 17.  $\cdot 2394$ .

### Examples XIV(a). [ Pages 157-160 ]

23.  $120^\circ$ . 24.  $A = 60^\circ$ . 29.  $A = 90^\circ$ ,  $B = 30^\circ$ ,  $C = 60^\circ$ .  
 39.  $\sqrt{\frac{y}{z} + \frac{z}{x} + \frac{x}{y}}$ . 40. 84.

### Examples XIV(b). [ Pages 166-168 ]

15.  $r = 4$ ;  $R = 8\frac{1}{2}$ .

**Examples XV(a).** [ Pages 172-173 ]

1.  $35^\circ 5' 49''$ .      2.  $102^\circ 1' 28''$ .      3.  $58^\circ 59' 38''$ .
4.  $104^\circ 30'$ ;  $46^\circ 36'$ ;  $28^\circ 54'$ .      5. (i)  $88^\circ 59' 40.9''$ .
- (ii)  $78^\circ 27' 46.86''$ .      6. (i)  $48^\circ 11' 29''$ ;  $58^\circ 24' 43''$ ;  $73^\circ 23' 54''$ .
- (ii)  $192^\circ 34' 24''$ .      7.  $A = 120^\circ$ ,  $B = 45^\circ$ ,  $C = 15^\circ$ .
8.  $A = 45^\circ$ ,  $B = 30^\circ$ ,  $C = 105^\circ$ .      9.  $A = 60^\circ$ ,  $B = 38^\circ 11'$ ,  $C = 81^\circ 49'$ .
10.  $A = 105^\circ$ ,  $B = 45^\circ$ ,  $C = 30^\circ$ .      11.  $(\sqrt{3}+1) : \sqrt{6} : (\sqrt{3}-1)$ .
13.  $\sqrt{5}+1 : \sqrt{5}-1$ .      14.  $3 : 4 : 5$ .

**Examples XV(b).** [ Pages 176-178 ]

1.  $B = 88^\circ 12' 46''$ ,  $C = 21^\circ 47' 12''$ .
2.  $B = 56^\circ 19' 46.3''$ ,  $C = 63^\circ 40' 13.7''$ .
3.  $A = 117^\circ 88' 45''$ ,  $B = 27^\circ 88' 45''$ .
4.  $A = 94^\circ 42' 54''$ ,  $B = 25^\circ 17' 6''$ .
5.  $B = 71^\circ 44' 29.5''$ ,  $C = 48^\circ 15' 30.5''$ .
6. (i)  $70^\circ 53' 36''$ ;  $49^\circ 6' 14''$ .      (ii)  $74^\circ 13' 50''$ ,  $35^\circ 16' 10''$ .
- (iii)  $A = 64^\circ 21'$ ,  $B = 77^\circ 25'$ ,  $c = 27.39$ .
7. (i)  $B = 78^\circ 17' 39.6''$ ,  $C = 49^\circ 36' 20.4''$ .
- (ii)  $116^\circ 33' 54''$ ;  $26^\circ 33' 54''$ .
8.  $A = B = 75^\circ$ ,  $C = 30^\circ$ ,  $b = 2\sqrt{6}$ .      9.  $\sqrt{6}$ ,  $15^\circ$ ,  $105^\circ$ .
10. (i)  $A = 45^\circ$ ,  $B = 75^\circ$ ,  $c = \sqrt{6}$ .      (ii)  $A = 30^\circ$ ,  $B = 90^\circ$ .
11.  $27.0375$ .      12.  $172.6436$  ft.      13.  $79.063$ .
14. (i)  $A = 31^\circ 20'$ ,  $b = 185$ ,  $c = 192$ .
- (ii)  $b = 18.46$ ,  $c = 37.16$ ,  $C = 70^\circ 30'$ .      (iii)  $b = 118.9$ ,  $c = 117.2$ .
15.  $C = 75^\circ$ ,  $a = c = 2\sqrt{3}+2$ .      16.  $C = 105^\circ$ ,  $a = \sqrt{2}$ ,  $c = \sqrt{3}+1$ .
17.  $72^\circ$ ,  $72^\circ$ ,  $36^\circ$ ; each side  $= \sqrt{5}+1$ .      18.  $8$ ,  $1$ .

**Examples XV(c).** [ Pages 184-185 ]

1. (i) One solution.      (ii) Ambiguous; two solutions.
- (iii) No solution.      (iv) One solution (right-angled triangle).
2. (i)  $C = 75^\circ$ ,  $A = 60^\circ$ ,  $a = \sqrt{6}$  }      (ii)  $60^\circ$ , or,  $120^\circ$ .
- or  $C = 105^\circ$ ,  $A = 30^\circ$ ,  $a = \sqrt{2}$  }
3.  $A = 45^\circ$ ,  $C = 75^\circ$ ,  $c = \sqrt{3}+1$ . (no ambiguity).      4. Impossible.
8.  $C = 58^\circ 56' 56.3''$  }      or,  $C = 121^\circ 3' 3.7''$  }
- $A = 87^\circ 48' 3.7''$  }       $A = 25^\circ 41' 56.3''$  }
9.  $B = 34^\circ 27'$ ,  $C = 100^\circ 33'$ .
10.  $A = 5^\circ 44' 21''$ .      11.  $A = 33^\circ 39' 34''$ ,  $B = 86^\circ 20' 26''$ .
12.  $A = 80^\circ 36'$ ,  $C = 64^\circ 14'$ ; or,  $A = 29^\circ 4'$ ,  $C = 115^\circ 46'$ .

## Miscellaneous Examples II. [Pages 186-188]

11. 4, 5, 6.                      14.  $B = 44^\circ 25' 39''$ , or,  $135^\circ 34' 21''$ .  
 21.  $\frac{1}{2} \{n\pi + \frac{1}{2}\pi - (a+b+c)\}$ .                      24.  $\frac{1}{2} (n\pi + \frac{1}{2}\pi)$ .

## Examples XVI. [Pages 213-214]

4.  $\theta = \frac{1}{2}\pi$ .                      5.  $x = 38^\circ 10'$  nearly.                      6.  $\frac{1}{2}\pi$ .                      7.  $-37$  nearly.  
 8. (i)  $x = 0$ .                      (ii)  $46^\circ 25'$  (nearly) and  $90^\circ$ .                      (iii)  $22\frac{1}{2}^\circ$  and  $112\frac{1}{2}^\circ$ .  
                     (iv)  $\frac{2}{3}\pi$ .                      (v)  $14^\circ$  nearly.                      (vi)  $1.19, 2.72, 4.92$ .  
                     (vii)  $1.16, 3.28, 4.95$ .                      (viii)  $\pm .82$ .                      (ix)  $.64$ .

## Examples XVII(a). [Pages 221-224]

4.  $\sqrt{2135}$  ft.                      14.  $17^\circ 27' 30''$ .                      18.  $18^\circ 26' 58''$  nearly.

## Examples XVII(b). [Pages 231-233]

1.  $-\cos\left(a + \frac{\pi}{2n}\right) / \sin \frac{\pi}{2n}$ .                      2. 0.  
 3.  $\frac{\sin\left\{a + \frac{1}{2}(n-1)(a+\pi)\right\} \sin \frac{1}{2}n(a+\pi)}{\sin \frac{1}{2}(a+\pi)}$ .  
 4.  $\frac{n}{2} + \frac{\sin n\theta}{2 \sin \theta} \cos(n+1)\theta$ .                      5.  $\frac{1}{4} \left( \frac{3 \sin^2 na}{\sin a} - \frac{\sin^2 3na}{\sin 3a} \right)$ .  
 6.  $(-1)^{n-1} \frac{\sin n\theta \sin(n+1)\theta}{2 \cos \theta}$ .  
 7.  $\frac{3}{8}n - \frac{1}{2} \frac{\sin n\pi}{\sin a} \cos(n+1)a + \frac{1}{8} \frac{\sin 2na}{\sin 2a} \cos 2(n+1)a$ .  
 8.  $\cos\left\{\theta + \frac{1}{2}(n-1)(\theta + \frac{1}{2}\pi)\right\} \frac{\sin \frac{1}{2}n(\theta + \frac{1}{2}\pi)}{\sin \frac{1}{2}(\theta + \frac{1}{2}\pi)}$ .  
 9.  $\frac{1}{4 \sin a} \left\{ (n+1) \sin 2a - \sin 2(n+1)a \right\}$ .  
 10.  $\frac{n}{2} \cos 2a + \frac{\cos 2(n+1)a \sin 2na}{2 \sin 2a}$ .                      11.  $\frac{1}{2}$ .                      12. 0.  
 13. 0.                      14.  $\sin na$ .                      16.  $\operatorname{cosec} a \{ \tan(n+1)a - \tan a \}$ .  
 17.  $\operatorname{cosec} \theta \{ \cot \theta - \cot(n+1)\theta \}$ .                      18.  $\frac{1}{2} \operatorname{cosec} a \{ \tan(n+1)a - \tan a \}$ .  
 19.  $\cot \theta \{ \cot \theta - \cot(n+1)\theta \} - n$ .                      20.  $\cot a - 2^n \cot 2^n a$ .  
 21.  $\frac{1}{2} \frac{\sin \frac{1}{2}n\theta}{\sin \frac{1}{2}\theta} \sin \frac{1}{2}(n+3)\theta - \frac{1}{4} \frac{\sin n\theta}{\sin \theta} \sin(n+3)\theta$ .  
 22.  $\frac{1}{2} (\tan 3^n x - \tan x)$ .                      23.  $\tan^{-1} \frac{n}{2+n}$ .                      24.  $\tan^{-1} \frac{n}{n+1}$ .  
 25.  $\tan^{-1} \frac{n}{n+1}$ .                      26.  $\frac{1}{2^{n-1}} \cot \frac{x}{2^{n-1}} - 2 \cot 2x$ .

27.  $\frac{1}{4} \left[ \frac{\sin \frac{1}{2} nx}{\sin \frac{1}{2} x} \cos \frac{1}{2} (n+3)x (1+2 \cos 2x) + \frac{\sin \frac{1}{2} nx}{\sin \frac{1}{2} x} \cos \frac{1}{2} (n+3)x \right]$ .
28.  $\frac{(n+1) \cos n\theta - n \cos (n+1)\theta - 1}{2(1 - \cos \theta)}$ . 29. (a)  $\frac{1}{2} n (n+1)$ . (b)  $n^2$ .
30.  $\cot x \tan (n+1)x - (n+1)$ ;  $\frac{1}{2} n (n+1)(n+2)$ .

### Examples XVII(c). [Pages 236-237]

1.  $(a^2 - b^2)^2 = ab$ . 2.  $a^2 \{(x+b)^2 + y^2\} = (x^2 + y^2 - b^2)^2$ .
3.  $(x+3y)^2 = xy^2 (x+2y)$ . 4.  $a^2 + b^2 = 1 + b^{\frac{2}{3}} - b^{\frac{4}{3}}$ .
5.  $(x+y)^{\frac{2}{3}} + (x-y)^{\frac{2}{3}} = 2$ . 6.  $(x^2 + y^2 + 2ax)^2 = 4a^2 (x^2 + y^2)$ .
7.  $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4^{\frac{2}{3}}$ . 8.  $x^{\frac{4}{3}} y^{\frac{2}{3}} - x^{\frac{2}{3}} y^{\frac{4}{3}} = 1$ . 9.  $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ .
10.  $\frac{2x}{a} = \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} \right) \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} - 3 \right)$ . 11.  $(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}$ .
12.  $\left( \frac{x}{a} + \frac{y}{b} \right)^{\frac{2}{3}} + \left( \frac{x}{a} - \frac{y}{b} \right)^{\frac{2}{3}} = 2$ . 13.  $x^{\frac{2}{3}} y^{\frac{2}{3}} (x^{\frac{2}{3}} + y^{\frac{2}{3}}) = 1$ .
14.  $3a - 2b = a^2$ . 15.  $\frac{x^2}{a} + \frac{y^2}{b} = a + b$ . 16.  $x^2 + y^2 - 2 \cos a = 2$ .
17.  $ab = (b-a) \tan a$ . 18.  $a+b = 2ab$ . 19.  $(ab-c)(a^2+b^2) = 2ab$ .

### Miscellaneous Examples III [Pages 254-263]

1.  $\frac{1}{18}\pi, \frac{1}{3}\pi, \frac{2}{3}\pi$ . 2.  $40^\circ, 60^\circ, 80^\circ, \frac{2}{3}\pi, \frac{1}{3}\pi, \frac{4}{3}\pi$ . 3.  $90^\circ, 22\frac{1}{2}^\circ, 67\frac{1}{2}^\circ$ .
4.  $1:36$ . 5.  $6\frac{1}{11}$  degrees. 6.  $54^\circ 46' 54''$ .
26.  $\frac{(t^2+1)(t^4+1)}{t^3}$ . 27. 1. 28.  $\frac{7}{15}$ . 29.  $\frac{1}{14}$ .
30.  $x = \frac{1}{2}\pi, y = \frac{1}{3}\pi$ . 31. (i) Possible. (ii) Impossible. (iii) Impossible. (iv) Possible.
32. (i)  $a^2 + b^2 = 2$ . (ii)  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$ . (iii)  $\left( \frac{x}{a} \right)^{\frac{2}{3}} + \left( \frac{y}{b} \right)^{\frac{2}{3}} = 1$ .
38. 341.5 ft. approximately. 39. 125 yds. 40. 173.2 ft.
41.  $90^\circ$ . 42. 273.2 ft. 43. 30 miles per hour.
44. height = 22.5 ft., distance = 38.97 ft. 45.  $1, \frac{2}{3}$ .
46.  $\frac{2}{3}\frac{2}{3}, \frac{2}{3}\frac{2}{3}$ . 47.  $\frac{2m\pi}{m^2 - n^2}$ . 48.  $-\frac{2}{3}$ . 49.  $\frac{2}{3}, -\frac{2}{3}, \frac{2}{3} \sqrt{5}$ .
50.  $\frac{1}{10} \sqrt{2}$ . 51.  $5 + 2 \sqrt{6}$ . 52.  $-\frac{2}{3}$ .
84.  $(2n+1) \frac{\pi}{2}$ , or,  $(2n+1) \pi \pm \frac{\pi}{8}$ , or,  $n\pi + (-1)^n \frac{\pi}{6}$ . 99. 107.2 ft.



# HIGHER SECONDARY EXAMINATION PAPERS

( B. O. S. E., W. B. )

1. (a) If  $A, B$  and  $A+B$  be all positive and acute, prove geometrically  $\cos(A+B) = \cos A \cos B - \sin A \sin B$ .

(b) Show that  $\cos 6^\circ \cos 42^\circ \cos 66^\circ \cos 78^\circ = \frac{1}{16}$ .

2. (a) Solve :  $\tan^2 x + \cot^2 x = 2$ .

(b) Solve :  $\sin^{-1} \frac{5}{x} + \sin^{-1} \frac{12}{x} = \frac{\pi}{2}$ .

3. (a) If  $\alpha + \beta = \gamma$ , show that

$$\cos^2 \alpha + \cos^2 \beta - 2 \cos \alpha \cos \beta \cos \gamma = \sin^2 \gamma.$$

(b) If  $\frac{\tan(\alpha + \beta - \gamma)}{\tan(\alpha - \beta + \gamma)} = \frac{\tan \gamma}{\tan \beta}$ , prove that either  $\sin(\beta - \gamma) = 0$

$$\text{or, } \sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0.$$

4. (a) If in a triangle

$$(a^2 + b^2) \sin(A - B) = (a^2 - b^2) \sin(A + B),$$

prove that the triangle is either isosceles or right-angled.

(b) Find the greatest angle of a triangle whose sides are 5, 6, 7 having given  $\log 6 = .7781513$ ,  $L \cos 39^\circ 14' = 9.8890644$ , diff. for  $60'' = .00011032$ .

5. (a) Draw the graph of  $y = \sin x + \cos x$  from  $x=0$  to  $x=2\pi$  having given :

	$10^\circ$	$20^\circ$	$30^\circ$	$40^\circ$	$50^\circ$	$60^\circ$	$70^\circ$	$80^\circ$
$\sin x$	.17	.34	.50	.64	.77	.87	.94	.98

(b) At each end of a horizontal line of length  $2a$ , the angular elevation of a certain peak is  $\theta$  and that at the middle point is  $\phi$ . Prove that the vertical height of the peak is

$$\frac{a \sin \theta \sin \phi}{\sin(\phi + \theta) \sin(\phi - \theta)}.$$

1. (a) If  $A, B, A+B$  be positive acute angles, prove geometrically that

$$\cos (A-B)=\cos A \cos B+\sin A \sin B.$$

(b) If  $A+B+C=\frac{\pi}{2}$ , prove that

$$\sin ^2 A+\sin ^2 B+\sin ^2 C=1-2 \sin A \sin B \sin C.$$

2. (a) Solve :  $2 \sin ^2 \theta+3 \cos \theta=0$ , ( $0^{\circ}<\theta<360^{\circ}$ ).

(b) If  $\sec \theta-\operatorname{cosec} \theta=\frac{4}{3}$ , prove that

$$\theta=\frac{1}{2} \sin ^{-1}\left(\frac{3}{4}\right).$$

3. (a) In a triangle  $ABC$ , prove that

$$\tan \frac{B-C}{2}=\frac{b-c}{b+c} \cot \frac{A}{2}.$$

(b) Two sides of a triangle are 80 and 100 feet and the included angle is  $60^{\circ}$ ; find the other angles having given

$$\log 3=.47712, L \tan 10^{\circ} 53' 36''=.928432.$$

4. (a) Prove that

$$\cos ^2 A+\cos ^2\left(A+\frac{\pi}{3}\right)+\cos ^2\left(A-\frac{\pi}{3}\right)=\frac{3}{2}.$$

(b) Prove that

$$2 \cot ^{-1} 5+\cot ^{-1} 7+2 \cot ^{-1} 8=\frac{\pi}{4}.$$

5. (a) Draw the graph of  $\cos x-\sin x$  between  $x=-\pi$  and  $x=\pi$ .

(b) A man standing on the bank of a river, observes that the angle subtended by a tower on the other bank just opposite to him is  $60^{\circ}$ . When he moves 60 ft. away from the bank in line with the tower he finds the angle subtended to be  $30^{\circ}$ ; find the height of the tower and the breadth of the river.

1. (a) If  $C, D$  and  $C+D$  be all positive acute angles, prove geometrically :  $\sin (C+D)=\sin C \cos D+\cos C \sin D$ .

(b) If  $A+B+C=\pi$  and  $\sin\left(A+\frac{C}{2}\right)=n\sin\frac{A}{2}$

prove that  $\tan\frac{A}{2}\cdot\tan\frac{B}{2}=\frac{n-1}{n+1}$

2. (a) Solve :—

$$\sin\theta + \sqrt{3}\cos\theta = \sqrt{2}$$

(b) Show that,

$$4(\cot^{-1}3 + \operatorname{cosec}^{-1}\sqrt{5}) = \pi$$

3. (a) If  $\tan\frac{\theta}{2} = \sqrt{\frac{1-e}{1+e}} \tan\frac{\phi}{2}$ ,

$$\text{show that } \cos\phi = \frac{\cos\theta - e}{1 - e\cos\theta}$$

(b) Prove that,  $\cos 36^\circ = \frac{1}{2}(\sqrt{5}+1)$

4. (a) Show that in any triangle  $ABC$ ,

$$a\cos A + b\cos B + c\cos C$$

$$= 4R\sin A \cdot \sin B \cdot \sin C.$$

where  $R$  is the circumradius of the triangle,

(b) The sides of a triangle are 2, 3 and 4. Find the greatest angle of the triangle, having given :

$$\log 2 = .3010300, \log 3 = .4771213,$$

$$L \tan 52^\circ 14' = 10.1108395,$$

$$L \tan 62^\circ 15' = 10.1111004.$$

5. (a) Draw the graph of  $y = 2\sin x^\circ + \cos x^\circ$ , for values of  $x$  between  $-30$  to  $+60$ , having given :

$x$ :	10	20	30	40	50	60	70	80
$\cos x^\circ$ :	.98	.94	.87	.77	.64	.50	.34	.17

6. (b) From the top  $A$  of a building, the angles of depression of the top  $B$  and bottom  $C$  of a lamp-post  $BC$  of height  $h$  and standing on the same horizontal plane are found to be  $\alpha$  and  $\beta$ . Find the height of the building and show that  $AB = \frac{h \cos \beta}{\sin(\beta - \alpha)}$

# GAUHATI UNIVERSITY INTER PAPERS

1. (a) Write down the expressions for  $\sin(A+B)$  and  $\cos(A+B)$  in terms of the trigonometrical ratios of  $A$  and  $B$ ; hence deduce that for  $\tan(A+B)$ .

(b) Solve:  $\sqrt{3} \cos \theta + \sin \theta = \sqrt{2}$  (Give general values).

2. (a) If  $A+B+C=\pi$ , prove that

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

(b) Show that

$$\tan^{-1} \sqrt{\frac{a(a+b+c)}{bc}} + \tan^{-1} \sqrt{\frac{b(a+b+c)}{ca}} + \tan^{-1} \sqrt{\frac{c(a+b+c)}{ab}} = \pi$$

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3. (a) In any triangle prove that

$$\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2}.$$

(b) The angles of a triangle, are in A. P., and the sides containing the mean angle are  $\sqrt{3}+1$  and  $\sqrt{3}-1$ ; find the other angles and the third side.

4. (a) Draw a neat graph of  $\cos x$  in  $-\pi < x < \pi$ .

(b) Show that

$$(i) \tan A = \frac{\sin 2A}{1 + \cos 2A};$$

$$(ii) \tan A + \cot A = 2 \operatorname{cosec} 2A.$$

1. (a) (i) Calculate  $\sin 22\frac{1}{2}^\circ$ .

(ii) Express  $2 \sin 11\theta \sin \theta$  as the difference of two terms.

(iii) If  $\tan A = \frac{4}{3}$ ,  $\tan B = \frac{1}{5}$ , find the smallest positive value of  $(A+B)$ .

(b) If  $A+B+C=\pi$ , prove that

$$\tan \frac{B}{2} \tan \frac{C}{2} + \tan \frac{C}{2} \tan \frac{A}{2} + \tan \frac{A}{2} \tan \frac{B}{2} = 1.$$

2. (a) In a triangle  $ABC$ , prove that

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

If  $a = 3\text{cm.}$ ,  $b = 4\text{cm.}$ ,  $c = 5\text{cm.}$ , find  $\tan \frac{C}{2}$ .

(b) Solve :  $\sin \theta + \sin 7\theta = \sin 4\theta$  (give general values).

3. (a) Express  $A \cos n\theta + B \sin n\theta$  in the form  $R \cos (n\theta - \alpha)$  and give the values of  $R$  and  $\alpha$ . State the amplitude and period of

$$3 \cos \theta + 4 \sin \theta.$$

(b) Show that

$$\tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{2} + \tan^{-1} 1 = \frac{\pi}{2}.$$

4. Either, (a) Tabulate the values of  $\sin x + \cos x$  as  $x$  varies from 0 to  $2\pi$  at intervals of  $\frac{\pi}{4}$ .

Or, Find the area of  $\triangle ABC$ , given that  $a = 13$ ,  $b = 14$ ,  $c = 15$ .

(b) Prove that

$$(i) \cos A = \frac{1 - \tan^2 \frac{A}{2}}{1 + \tan^2 \frac{A}{2}}$$

$$(ii) \sqrt{\frac{1 + \sin A}{1 - \sin A}} = \sec A + \tan A.$$

## PATNA UNIVERSITY QUESTIONS

1. (a) Show that  $\cos^2 A \cos 3A + \sin^2 A \sin 3A = \cos^2 2A$ .

(b) If  $x \sin^2 \theta + y \cos^2 \theta = \sin \theta \cos \theta$ , and  $x \sin \theta - y \cos \theta = 0$ , show that  $x^2 + y^2 = 1$ .

2. (a) Establish the formula

$$\cos B - \cos A = 2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}.$$

(b) Prove that  $\cos^2 A + \cos^2 B + \cos^2 C - 2 \cos A \cos B \cos C = 1$ , if  $A+B=C$ .

3. (a) Prove that in a triangle,  $\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$ .

(b)  $\frac{a^2 \sin(B-C)}{\sin B + \sin C} + \frac{b^2 \sin(C-A)}{\sin C + \sin A} + \frac{c^2 \sin(A-B)}{\sin A + \sin B} = 0$ .

4. (a) Draw the graph of  $y = \sin x + \cos x$  as  $x$  ranges from 0 to  $\pi$ .

(b) Prove that  $\cot A + \cot B + \cot C = \cot A \cot B \cot C$ , if  $A+B+C = \frac{1}{2}\pi$ .

5. (a) Prove that  $\log_b n = \log_a n \times \log_b a$ .

(b) To determine the breadth  $AB$  of a canal an observer places himself at  $C$  in the straight line  $AB$  produced through  $C$ , and then walks 100 yards at right angles to the line. He then finds that  $AB$  and  $BC$  subtend angles  $15^\circ$  and  $25^\circ$  at his eyes. Find the breadth of the canal, given  $L \cos 25^\circ = 9.9572757$ ;  $L \cos 40^\circ = 9.8842540$ ;  $L \cos 75^\circ = 9.4129962$ ;  $\log 37279 = 4.5714643$ ;  $\log 3728 = 3.5714759$ .

1. (a) Evaluate  $\sin 18^\circ$ .

(b) If  $\sec(\phi + \alpha) + \sec(\phi - \alpha) = 2 \sec \phi$ , prove that

$$\cos \phi = \sqrt{2} \cos \frac{\alpha}{2}.$$

2. (a) If  $A+B+C = \pi$ , prove that

$$\cos \frac{A}{2} + \cos \frac{B}{2} + \cos \frac{C}{2} = 4 \cos \frac{\pi-A}{4} \cos \frac{\pi-B}{4} \cos \frac{\pi-C}{4}.$$

(b) Draw the graph of  $y = \tan x$  from  $x=0$  to  $x=2\pi$ .

3. In a triangle  $ABC$ , prove that

$$(i) \cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}.$$

(ii)  $\cot A, \cot B, \cot C$  are in A.P., if  $a^2, b^2, c^2$  are in A.P.

4. Two sides of a triangle are in the ratio of 9 to 7, and the included angle is  $64^\circ 12'$ ; find the other angles, having given  $\log 2 = .3010300$ ,  $L \tan 57^\circ 54' = 10.2025255$ ,  $L \tan 11^\circ 16' = 9.2993216$ ,  $L \tan 11^\circ 17' = 9.2999804$ .

5. A flagstaff  $PN$  stands vertically on level ground. A base  $XY$  is measured at right angles to  $XN$ , the points  $X, Y, N$  being in the same horizontal plane, and the angle  $PXN$  and  $PYN$  are found to be  $\alpha$  and  $\beta$  respectively. Prove that the height of the flagstaff is

$$\frac{\sin \alpha \sin \beta}{\sqrt{\sin (\alpha - \beta) \sin (\alpha + \beta)}} XY.$$

1. (a) If  $\frac{x}{\tan (\theta + \alpha)} = \frac{y}{\tan (\theta + \beta)} = \frac{z}{\tan (\theta + \gamma)}$ , prove that

$$\frac{x+y}{x-y} \sin^2 (\alpha - \beta) + \frac{y+z}{y-z} \sin^2 (\beta - \gamma) + \frac{z+x}{z-x} \sin^2 (\gamma - \alpha) = 0.$$

(b) Prove that  $\cot A + \cot (60^\circ + A) + \cot (120^\circ + A) = 3 \cot 3A$ .

2. (a) If  $A + B + C = 180^\circ$  and  $\sin \left( A + \frac{C}{2} \right) = n \sin \frac{C}{2}$ , show that

$$\tan \frac{A}{2} \tan \frac{B}{2} = \frac{n-1}{n+1}.$$

(b) If  $A + B + C = 180^\circ$ , prove that

$$\sin^2 A + \sin^2 B + \sin^2 C - 2 \cos A \cos B \cos C = 2.$$

3. In a triangle, Prove that

$$(i) \sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}.$$

$$(ii) (b^2 - c^2) \cot A + (c^2 - a^2) \cot B + (a^2 - b^2) \cot C = 0.$$

4. (a) Draw the graph of  $y = \sin x$  from  $x = 0$  to  $x = \pi$ , and from the graph find the angle whose sine is .7.

(b) If  $a = 70$ ,  $b = 85$ ,  $C = 86^\circ 52' 12''$ , find the other angles, having given  $\log 3 = .4771213$ ,  $L \cot 18^\circ 26' 6'' = 10.4771213$ .

5. A flagstaff is on the top of a tower which stands on a level plane. At a certain point in the plane the tower subtends an angle  $\alpha$ , and the flagstaff an angle  $\beta$ . At another point  $a'$  ft. nearer the base of the tower, the flagstaff again subtends the angle  $\beta$ . Show that the height of the tower is  $\frac{a \tan \alpha}{1 - \tan \alpha \tan (\alpha + \beta)}$ .

# ALLAHABAD UNIVERSITY QUESTIONS

1. (a) Prove that  $\cot \frac{\sin A + \sin B}{\sin A - \sin B} = \tan \frac{A+B}{2} \cot \frac{A-B}{2}$ .

(b) Prove that  $\cot \left( \frac{\pi}{4} + \theta \right) \times \cot \left( \frac{\pi}{4} - \theta \right) = 1$ .

2. (a) Prove that  $\frac{\sin \theta + \sin 2\theta}{1 + \cos \theta + \cos 2\theta} = \tan \theta$ .

(b) Solve the equation  $\sqrt{3} \cos \theta + \sin \theta = \sqrt{2}$ .

3. (a) Prove that  $\tan \left( 45^\circ + \frac{A}{2} \right) = \sqrt{\frac{1 + \sin A}{1 - \sin A}} = \sec A + \tan A$ .

(b) If  $A + B + C = 180^\circ$ , prove that

$$\sin^2 A + \sin^2 B - \sin^2 C = 2 \sin A \sin B \cos C.$$

4. (a) Prove the formula  $r = \frac{S}{s}$ , where the letters have their usual meanings.

(b) Prove that in any triangle,  $(r_1 - r)(r_2 - r)(r_3 - r) = 4Rr^3$ .

5. The sides of a triangle are 32, 40, and 66 feet. Find the angle opposite the greatest side, having given

$$\log 3 = .47712, \log 69 = 1.83885, \log 37 = 1.56820,$$

$$\log 29 = 1.46240, L \cot 66^\circ 10' = 9.64517, L \cot 66^\circ 20' = 9.64175.$$

6. (a) If  $A + B + C = 2R$ , prove that

$$\sin (S - A) + \sin (S - B) + \sin (S - C) - \sin S$$

$$= 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}.$$

(b) In any triangle  $ABC$ , prove that

$$(b + c - a) \left( \cot \frac{B}{2} + \cot \frac{C}{2} \right) = 2a \cot \frac{A}{2}.$$

7. At each end of a horizontal base of length  $2a$  it is found that the angular height of a certain peak is  $\theta$ , and that at the middle point it is  $\phi$ . Prove that the vertical height of the peak is

$$\frac{a \sin \theta \sin \phi}{\sqrt{\sin (\phi + \theta) \sin (\phi - \theta)}}.$$



1. (a) Prove that

$$\frac{\cos A + \cos B}{\cos B - \cos A} = \cot \frac{A+B}{2} \cot \frac{A-B}{2}.$$

- (b) Solve the equation
- $4 \cos^2 \theta + \sqrt{3} = 2(\sqrt{3} + 1) \cos \theta$
- .

2. (a) If
- $\tan \alpha = \frac{4}{5}$
- , and
- $\tan \beta = \frac{1}{11}$
- , prove that

$$\alpha + \beta = \frac{\pi}{4}.$$

- (b) Show that

$$2 \cos \frac{\pi}{18} + \cos \frac{9\pi}{18} + \cos \frac{3\pi}{18} + \cos \frac{5\pi}{18} = 0.$$

3. (a) Find the value of
- $\cos 36^\circ$
- .

- (b) Prove that

$$\cot A = \frac{1}{2} \left( \cot \frac{A}{2} - \tan \frac{A}{2} \right).$$

4. (a) Prove the formula
- $R = \frac{abc}{4S}$
- , where the letters have their usual meanings.

- (b) Prove that

$$\frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab} = \frac{1}{2Rr}.$$

5. In the triangle
- $ABC$
- ,
- $b=16$
- ,
- $c=25$
- and the angle
- $B=33^\circ 15'$
- . Find the remaining angles if

$$\log 2 = .30103,$$

$$L \sin 33^\circ 15' = 9.7390129,$$

$$L \sin 58^\circ 56' = 9.9327616, \text{ and}$$

$$L \sin 58^\circ 57' = 9.9328876.$$

6. (a) If
- $A+B+C=2S$
- , prove that

$$\sin(S-A) \sin(S-B) + \sin S \sin(S-C) = \sin A \sin B.$$

- (b) If the sides of a triangle be in arithmetical progression, prove that so also are the cotangents of half the angles.

7. At a distance
- $a$
- from the foot
- $A$
- of a tower
- $AB$
- , of known height
- $b$
- , a flagstaff
- $BC$
- on the top of the tower and the tower both subtend equal angles. Find the height of the flagstaff.

1. (a) Prove that

$$\frac{\sin 7A - \sin 5A}{\cos 5A + \cos 7A} = \tan A.$$

- (b) Prove that

$$\sin A = 2 \tan \frac{A}{2} \bigg/ \left(1 + \tan^2 \frac{A}{2}\right).$$

2. (a) Prove that
- $\tan 2A = (\sec 2A + 1) \sqrt{\sec^2 A - 1}$
- .

- (b) Solve the equation
- $\sin \theta + \sin 3\theta + \sin 5\theta = 0$
- .

3. (a) Find the value of
- $\sin 22\frac{1}{2}^\circ$
- .

- (b) Prove that

$$\sin^2\left(\frac{\pi}{8} + \frac{A}{2}\right) - \sin^2\left(\frac{\pi}{8} - \frac{A}{2}\right) = \frac{1}{\sqrt{2}} \sin A.$$

4. (a) Prove the formula

$$r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2},$$

where the letters have their usual meanings.

- (b) Prove that

$$r_1 r_2 r_3 = r^3 \cot^2 \frac{A}{2} \cot^2 \frac{B}{2} \cot^2 \frac{C}{2}.$$

5. If the lengths of the greatest and least sides of a triangle be 24 and 16 feet respectively and the angle between them be  $60^\circ$ , find the length of the third side and the remaining angles, given  $\log 2 = .30103$ ,  $\log 3 = .4771213$ , and  $L \tan 19^\circ 6' = 9.5394217$ , diff. for  $1' = 4084$ .

6. (a) If
- $A + B + C = 2S$
- , prove that

$$\begin{aligned} \cos^2 S + \cos^2(S - A) + \cos^2(S - B) + \cos^2(S - C) \\ = 2 + 2 \cos A \cos B \cos C. \end{aligned}$$

- (b) In any triangle
- $ABC$
- , prove that

$$a \sin(B - C) + b \sin(C - A) + c \sin(A - B) = 0.$$

7. A tower, more than 100 feet high, consists of two parts, the lower being one-third of the whole. At a point in a horizontal plane through the foot of the tower and 40 feet from it, the upper part subtends an angle whose tangent is  $\frac{1}{2}$ . Find the height of the tower.

# BENARES HINDU UNIVERSITY QUESTIONS

1. Expand the determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ \sin A & \sin B & \sin C \\ \cos A & \cos B & \cos C \end{vmatrix}$$

Hence or otherwise prove that for any  $\triangle ABC$ ,

$$a \sin (B-C) + b \sin (C-A) + c \sin (A-B) = 0.$$

2. (a) Prove that

$$2 \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{7} = \frac{\pi}{4}.$$

(b) Solve the equation  $\tan \theta = \cot \theta$ .

3. Prove the following :

$$(i) \frac{\sec 8A - 1}{\sec 4A - 1} = \tan 8A \cot 2A.$$

$$(ii) \tan 6^\circ \tan 42^\circ \tan 66^\circ \tan 78^\circ = 1.$$

$$(iii) \sin (A+B) \sin (A-B) = \sin^2 A - \sin^2 B.$$

4. (a) Obtain the radius of the in-circle of a triangle in terms of the lengths of its sides.

(b) If  $r$  and  $R$  denote respectively the radii of the inscribed and circumscribed circles of a triangle  $ABC$ , prove that

$$\frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab} = \frac{1}{2rR}.$$

5. (a) In any triangle  $ABC$ , prove that

$$\tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2}.$$

(b) In a triangle  $ABC$ ,  $a=540$  yards,  $b=420$  yards,  $\angle C=50^\circ 6'$ . Find the unknown angles, having given—

$$\log 2 = .30103$$

$$L \tan 26^\circ 3' = 9.6891480$$

$$L \tan 14^\circ 20' = 9.4074189$$

$$L \tan 14^\circ 21' = 9.4079453$$

6. The angle of elevation of a cloud from a point  $h$  feet above a lake is  $\alpha$  and the angle of depression of its reflection in the lake is  $\beta$ . Prove that the height of the cloud above the lake is

$$h \frac{\sin(\beta + \alpha)}{\sin(\beta - \alpha)}.$$

7. (a) Prove that  $\sin^2 A + \sin^2 B - \sin^2 C = 2 \sin A \sin B \cos C$ , where  $A + B + C = 180^\circ$ .

(b) Solve the equation

$$\sqrt{3} \cos \theta + \sin \theta = \sqrt{2}.$$

1. (a) If an angle subtended by an arc of length  $l$  at the centre of a circle of radius  $r$  be taken as a unit, and three angles  $A^\circ$ ,  $B^\circ$  and  $C$  radians expressed in that unit be  $x$ ,  $y$ ,  $z$  respectively, show that

$$x : y : z = \frac{A\pi}{18} : \frac{B\pi}{20} : 10C.$$

(b) Prove that

$$\cos(A - B) = \cos A \cos B + \sin A \sin B.$$

2. Solve :

(i)  $\tan(\pi \cot \theta) = \cot(\pi \tan \theta).$

(ii)  $\sin 2\theta = \cot 3\theta.$

(iii)  $\sqrt{3} \sin \theta - \cos \theta = \sqrt{2}.$

3. Prove that

$$\begin{aligned} \text{(i)} \quad & \tan(\tan^{-1}x + \tan^{-1}y + \tan^{-1}z) \\ &= \cot(\cot^{-1}x + \cot^{-1}y + \cot^{-1}z). \end{aligned}$$

$$\text{(ii)} \quad \left(\frac{1}{r} - \frac{1}{r_1}\right)\left(\frac{1}{r} - \frac{1}{r_2}\right)\left(\frac{1}{r} - \frac{1}{r_3}\right) = \frac{16R}{r^2(a+b+c)^2},$$

where the symbols  $r$ ,  $r_1$ ,  $r_2$ ,  $r_3$  and  $R$  have their usual meanings.

4. (a) Prove that

$$16 \cos \frac{2\pi}{15} \cos \frac{4\pi}{15} \cos \frac{8\pi}{15} \cos \frac{14\pi}{15} = 1.$$

(b) In any triangle  $ABC$ , prove that

$$\begin{aligned} & a^2 (\cos^2 B - \cos^2 C) + b^2 (\cos^2 C - \cos^2 A) \\ & \quad + c^2 (\cos^2 A - \cos^2 B) = 0. \end{aligned}$$

5. (a) Prove that

$$\sin 4A = 4 \sin A \cos^3 A - 4 \cos A \sin^3 A.$$

(b) If the angles of a triangle are in A.P. and the lengths of the greatest and least sides be 24 and 16 feet respectively, find the length of the third side and the angles, given—

$$\log 2 = \cdot 3010300,$$

$$\log 8 = \cdot 4771213$$

$$L \tan 19^\circ 6' = 9.5394287,$$

$$\text{diff. for } 1' = 4084.$$

1. Prove the following :

$$(i) \frac{1}{\sec A - \tan A} = \sec A + \tan A.$$

$$(ii) \sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ = \frac{1}{16}.$$

$$(iii) \tan A + \tan B + \tan C = \tan A \tan B \tan C, \text{ where } A + B + C = 180^\circ.$$

$$2. (a) \text{ Prove that } \sin^{-1} \frac{3}{5} + \sin^{-1} \frac{4}{5} = \sin^{-1} \frac{7}{25}.$$

$$(b) \text{ Solve } \tan \theta + \tan 2\theta + \sqrt{3} \tan \theta \tan 2\theta = \sqrt{3}.$$

3. (a) Discuss the 'ambiguous case' in the solution of triangles.

(b) In the ambiguous case, given  $a$ ,  $b$  and  $A$ , prove that the difference between the two values of  $c$  is  $2\sqrt{a^2 - b^2 \sin^2 A}$ .

$$4. (a) \text{ In any triangle } ABC, \text{ prove that } \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

$$(b) \text{ Prove that } r_1 + r_2 + r_3 - r = 4R.$$

5. An object is observed at three points  $A$ ,  $B$ ,  $C$  lying in a horizontal straight line which passes directly underneath the object. The angular elevation at  $B$  is twice that at  $A$ , and at  $C$  it is three times that at  $A$ . If  $AB = a$ ,  $BC = b$ , show that the height of the object is  $\frac{a}{2b} \sqrt{(a+b)(3b-a)}$ .

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**TABLES OF LOGARITHMS, NATURAL SINES,  
NATURAL TANGENTS, LOGARITHMIC SINES,  
LOGARITHMIC TANGENTS ETC.**

TABLE I  
LOGARITHMS OF NUMBERS

	0	1	2	3	4	5	6	7	8	9	Mean Differences									
10	00000	00432	00860	01284	01703	02119	02531	02938	03342	03743	42	83	125	166	208	248	290	331	373	
11	04139	04593	04922	05308	05690	06070	06446	06819	07188	07555	38	76	114	152	190	227	265	302	340	
12	07918	08279	08686	09091	09542	09891	10037	10380	10721	11059	35	70	105	140	175	209	243	278	313	
13	11394	11727	12057	12385	12710	13038	13364	13672	13988	14301	32	65	97	129	163	198	225	258	290	
14	14613	14922	15229	15534	15836	16137	16435	16732	17026	17319	30	60	90	120	150	180	210	240	270	
15	17609	17898	18184	18469	18752	19033	19312	19590	19866	20140	28	56	84	112	140	168	196	224	252	
16	20413	20683	20952	21219	21484	21748	22011	22273	22531	22789	26	53	79	105	132	158	184	210	237	
17	23045	23300	23553	23805	24055	24304	24551	24797	25042	25285	25	50	74	99	124	149	174	199	223	
18	25527	25768	26007	26245	26482	26717	26951	27184	27416	27646	23	47	70	94	117	141	164	188	211	
19	27875	28108	28330	28556	28780	29003	29226	29447	29667	29885	22	45	67	89	111	134	156	178	201	
20	30103	30390	30635	30750	30963	31175	31387	31597	31806	32016	21	42	64	85	106	127	148	170	191	
21	32232	32428	32634	32838	33041	33244	33445	33646	33846	34044	20	40	61	81	101	121	141	162	183	
22	34243	34439	34635	34830	35026	35218	35411	35603	35793	35984	19	39	58	77	97	116	135	154	174	
23	36173	36361	36549	36736	36922	37107	37291	37475	37658	37840	19	37	56	74	93	111	130	148	167	
24	38021	38203	38382	38561	38739	38917	39094	39270	39445	39620	18	36	53	71	89	107	124	142	160	
25	39794	39967	40140	40312	40483	40654	40824	40993	41162	41330	17	34	51	68	85	102	119	136	153	
26	41497	41664	41830	41996	42160	42325	42488	42651	42813	42975	16	33	49	66	82	98	115	131	148	
27	43136	43297	43457	43616	43775	43933	44091	44248	44404	44560	16	32	47	63	79	95	111	126	142	
28	44716	44871	45025	45179	45332	45484	45637	45788	45939	46090	15	30	46	61	76	91	106	122	137	
29	46240	46399	46558	46697	46835	46982	47139	47276	47422	47567	15	29	44	59	74	88	103	118	132	

TABLE I ]

## LOGARITHMS OF NUMBERS

III

30	47713	47857	48001	48144	48287	48430	48572	48714	48856	48996	14	29	43	57	72	86	100	114	129
31	49186	49276	49415	49554	49693	49831	49969	50106	50243	50379	14	28	42	55	69	83	97	110	125
32	50615	50651	50786	50920	51055	51188	51322	51455	51587	51720	13	27	40	54	67	80	94	107	121
33	51851	51983	52114	52244	52375	52504	52638	52763	52892	53020	13	26	39	52	65	78	91	104	117
34	53148	53276	53403	53529	53656	53782	53908	54033	54158	54283	13	25	38	50	63	76	88	101	113
35	54407	54531	54654	54777	54900	55023	55145	55267	55388	55509	12	24	37	49	61	73	86	98	110
36	55630	55751	55871	55991	56110	56239	56348	56467	56585	56703	12	24	36	48	60	71	83	95	107
37	56820	56937	57064	57171	57287	57403	57519	57634	57749	57864	12	23	35	46	58	70	81	93	104
38	57978	58092	58206	58320	58433	58546	58659	58771	58883	58995	11	23	34	45	57	68	79	90	102
39	59106	59218	59329	59439	59550	59660	59770	59879	59988	60097	11	22	33	44	55	66	77	88	99
40	60206	60314	60423	60531	60638	60746	60853	60959	61066	61173	11	21	32	43	54	64	75	86	97
41	61278	61384	61490	61595	61700	61805	61909	62014	62118	62221	10	21	31	42	52	63	73	84	94
42	62325	62428	62531	62634	62737	62839	62941	63043	63144	63246	10	20	31	41	51	61	71	82	92
43	63347	63448	63548	63649	63749	63849	63949	64048	64147	64246	10	20	30	40	50	60	70	80	90
44	64345	64444	64543	64642	64738	64836	64933	65031	65128	65225	10	20	29	39	49	59	68	78	88
45	65321	65418	65514	65610	65706	65801	65896	65992	66087	66181	10	19	29	38	48	57	67	76	86
46	66276	66370	66464	66558	66652	66745	66839	66932	67025	67117	9	19	28	37	47	56	65	75	84
47	67310	67302	67394	67486	67578	67669	67761	67852	67943	68034	9	18	27	37	46	55	64	73	82
48	68124	68215	68305	68395	68485	68574	68664	68753	68842	68931	9	18	27	36	45	53	62	71	80
49	69020	69108	69197	69285	69373	69461	69548	69636	69723	69810	9	18	26	35	44	53	61	70	79
50	69897	69984	70070	70157	70243	70329	70415	70501	70586	70672	9	17	26	34	43	52	60	69	77
51	70757	70842	70927	71012	71096	71181	71265	71349	71433	71517	8	17	25	34	42	51	59	67	76
52	71600	71684	71767	71850	71933	72016	72099	72181	72263	72346	8	17	25	33	42	50	58	66	75
53	72428	72509	72591	72673	72754	72835	72916	72997	73078	73159	8	16	24	32	41	49	57	65	73
54	73239	73320	73400	73480	73560	73640	73719	73799	73878	73957	8	16	24	32	40	48	56	64	72
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9



## TRIGONOMETRY

## LOGARITHMS OF NUMBERS

	0	1	2	3	4	5	6	7	8	9
55	74086	74116	74194	74273	74351	74429	74507	74586	74663	74741
56	74819	74896	74974	75051	75128	75205	75282	75358	75435	75511
57	75597	75664	75740	75816	75891	75967	76043	76118	76193	76268
58	76343	76418	76492	76567	76641	76716	76790	76864	76938	77012
59	77085	77169	77252	77335	77418	77501	77584	77667	77750	77833
60	77815	77897	77980	78062	78144	78226	78308	78390	78472	78554
61	78638	78720	78802	78884	78966	79048	79130	79212	79294	79376
62	79399	79481	79563	79645	79727	79809	79891	79973	80055	80137
63	80258	80340	80422	80504	80586	80668	80750	80832	80914	81000
64	81081	81163	81245	81327	81409	81491	81573	81655	81737	81819
65	81901	81983	82065	82147	82229	82311	82393	82475	82557	82639
66	82701	82783	82865	82947	83029	83111	83193	83275	83357	83439
67	83501	83583	83665	83747	83829	83911	83993	84075	84157	84239
68	84301	84383	84465	84547	84629	84711	84793	84875	84957	85039
69	85101	85183	85265	85347	85429	85511	85593	85675	85757	85839
70	85901	85983	86065	86147	86229	86311	86393	86475	86557	86639
71	86701	86783	86865	86947	87029	87111	87193	87275	87357	87439
72	87501	87583	87665	87747	87829	87911	87993	88075	88157	88239
73	88301	88383	88465	88547	88629	88711	88793	88875	88957	89039
74	89101	89183	89265	89347	89429	89511	89593	89675	89757	89839

TABLE I.]

## LOGARITHMS OF NUMBERS

	75	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99
	87606	87564	87623	87679	87737	87795	87852	87910	87967	88024	88081	88138	88195	88252	88309	88366	88423	88480	88536	88593	88649	88705	88762	88818	88874
	889309	88986	88931	89376	89432	89487	89543	89597	89653	89708	89763	89818	89873	89927	89982	90037	90091	90146	90200	90255	90309	90363	90417	90472	90526
	90580	90634	90687	90741	90795	91116	91169	91223	91275	91328	91645	91698	91751	91803	91855	92169	92221	92273	92324	92376	92686	92737	92788	92840	92891
	93197	93247	93298	93349	93399	93702	93752	93802	93852	93902	94201	94250	94300	94349	94399	94694	94743	94792	94841	94890	95182	95231	95279	95328	95376
	95624	95673	95713	95761	95809	95856	96142	96190	96237	96284	96567	96614	96661	96708	96755	97081	97128	97174	97220	97267	97543	97589	97635	97681	97727
	97773	97818	97864	97909	97955	98000	98045	98091	98137	98182	98463	98498	98543	98588	98632	98900	98945	98989	99034	99078	99344	99388	99433	99476	99520
	99564	99607	99651	99695	99739	99782	99826	99870	99913	99957															
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9						

TABLE II  
NATURAL SINES

	Mean Differences																
	0'	10'	20'	30'	40'	50'	60'	1'	2'	3'	4'	5'	6'	7'	8'	9'	
0°	0.00000	0.00291	0.00582	0.00873	0.01164	0.01454	0.01745	89°	29	58	87	116	145	175	204	233	262
1°	0.01745	0.02036	0.02327	0.02618	0.02908	0.03199	0.03490	88°	29	58	87	116	145	175	204	233	262
2°	0.03490	0.03781	0.04071	0.04362	0.04653	0.04943	0.05234	87°	29	58	87	116	145	175	204	233	262
3°	0.05234	0.05524	0.05814	0.06105	0.06395	0.06685	0.06976	86°	29	58	87	116	145	174	203	232	261
4°	0.06976	0.07266	0.07556	0.07846	0.08136	0.08426	0.08716	85°	29	58	87	116	145	174	203	232	261
5°	0.08716	0.09005	0.09295	0.09585	0.09874	0.10164	0.10453	84°	29	58	87	116	145	174	203	232	261
6°	0.10453	0.10743	0.11031	0.11320	0.11609	0.11898	0.12187	83°	29	58	87	116	145	174	203	232	261
7°	0.12187	0.12476	0.12764	0.13053	0.13341	0.13629	0.13917	82°	29	58	87	116	145	173	203	231	260
8°	0.13917	0.14205	0.14493	0.14781	0.15069	0.15356	0.15643	81°	29	58	86	115	144	173	202	230	259
9°	0.15643	0.15931	0.16218	0.16505	0.16792	0.17078	0.17365	80°	29	57	86	115	144	172	201	229	258
10°	0.17365	0.17651	0.17937	0.18224	0.18509	0.18795	0.19081	79°	29	57	86	115	144	172	201	229	258
11°	0.19081	0.19366	0.19652	0.19937	0.20222	0.20507	0.20791	78°	29	57	86	114	143	171	200	228	257
12°	0.20791	0.21076	0.21360	0.21644	0.21928	0.22212	0.22495	77°	28	57	85	114	142	170	199	227	256
13°	0.22495	0.22778	0.23062	0.23345	0.23627	0.23910	0.24192	76°	28	57	85	113	141	170	198	226	255
14°	0.24192	0.24474	0.24756	0.25038	0.25320	0.25601	0.25882	75°	28	56	85	113	141	169	197	225	254
15°	0.25882	0.26163	0.26443	0.26724	0.27004	0.27284	0.27564	74°	28	56	84	112	140	168	196	224	252
16°	0.27564	0.27843	0.28123	0.28402	0.28680	0.28959	0.29237	73°	28	56	84	112	140	167	195	223	251
17°	0.29237	0.29515	0.29793	0.30071	0.30348	0.30625	0.30902	72°	28	56	83	111	139	166	194	222	250
18°	0.30902	0.31178	0.31454	0.31730	0.32006	0.32283	0.32557	71°	28	55	83	110	138	166	193	221	248
19°	0.32557	0.32833	0.33106	0.33381	0.33655	0.33929	0.34202	70°	27	55	82	110	137	164	192	219	247



## NATURAL SINES

	Mean Differences																	
	1'	2'	3'	4'	5'	6'	7'	8'	9'									
	20	41	61	82	102	122	143	163	184	44°								
	20	40	60	80	100	120	140	160	180	43°								
	20	39	59	78	98	118	138	157	177	42°								
	19	39	58	77	96	116	135	154	173	41°								
	19	38	57	76	95	113	132	151	170	40°								
	19	37	56	74	93	111	130	148	167	39°								
	18	36	54	72	91	109	127	145	163	38°								
	18	35	53	71	89	106	124	142	159	37°								
	17	35	52	69	87	104	121	138	156	36°								
	17	34	51	68	85	101	118	135	152	35°								
	16	33	49	66	82	99	115	132	148	34°								
	16	32	48	64	80	96	112	128	144	33°								
	16	31	47	63	78	94	110	125	141	32°								
	15	30	46	61	76	91	106	122	137	31°								
	15	30	44	59	74	89	103	118	133	30°								
	14	29	43	57	72	86	100	114	129	29°								
	14	28	42	55	69	83	97	111	125	28°								
	13	27	40	54	67	81	94	108	121	27°								
	13	26	39	52	65	78	91	104	117	26°								
	13	25	38	50	63	75	88	100	113	25°								



TABLE III  
NATURAL TANGENTS

	0'	10'	20'	30'	40'	50'	60'		1'	2'	3'	4'	5'	6'	7'	8'	9'
0°	0.00000	0.00291	0.00582	0.00873	0.01164	0.01455	0.01746	89°	29	58	87	116	146	175	204	233	262
1°	.01746	.02037	.02328	.02619	.02910	.03201	.03492	88°	29	58	87	116	146	175	204	233	262
2°	.03492	.03783	.04075	.04366	.04658	.04949	.05241	87°	29	58	87	116	146	175	204	233	262
3°	.05241	.05533	.05824	.06116	.06408	.06700	.06993	86°	29	58	88	117	146	175	204	234	263
4°	.06993	.07285	.07578	.07870	.08163	.08456	.08749	85°	29	58	88	117	146	175	204	234	263
5°	.08749	.09042	.09335	.09629	.09923	.10216	.10510	84°	29	59	88	118	147	176	206	235	265
6°	.10510	.10805	.11099	.11394	.11688	.11983	.12278	83°	29	59	88	118	147	176	206	235	265
7°	.12278	.12574	.12869	.13165	.13461	.13758	.14054	82°	30	59	89	118	148	178	207	237	266
8°	.14054	.14351	.14648	.14945	.15243	.15540	.15838	81°	30	59	89	119	149	178	208	238	267
9°	.15838	.16137	.16435	.16734	.17033	.17333	.17633	80°	30	60	90	120	150	179	209	239	269
10°	.17633	.17933	.18233	.18534	.18835	.19136	.19438	79°	30	60	90	120	151	181	211	241	271
11°	.19438	.19740	.20042	.20345	.20648	.20952	.21256	78°	30	61	91	121	152	182	212	242	273
12°	.21256	.21560	.21864	.22169	.22475	.22781	.23087	77°	31	61	92	122	153	183	214	244	275
13°	.23087	.23393	.23700	.24008	.24316	.24624	.24933	76°	31	62	92	123	154	185	216	246	277
14°	.24933	.25242	.25552	.25862	.26172	.26483	.26795	75°	31	62	93	124	155	186	217	248	279
15°	.26795	.27107	.27419	.27732	.28046	.28360	.28675	74°	31	63	94	125	157	188	219	250	282
16°	.28675	.28990	.29305	.29621	.29938	.30255	.30573	73°	32	63	95	126	158	190	221	253	285
17°	.30573	.30891	.31210	.31530	.31850	.32171	.32492	72°	32	64	96	128	160	192	224	256	288
18°	.32492	.32814	.33136	.33460	.33783	.34108	.34433	71°	32	65	97	129	162	194	226	259	291
19°	.34433	.34758	.35085	.35412	.35740	.36068	.36397	70°	33	65	98	131	164	196	229	262	294

TABLE III

## NATURAL TANGENTS

XI

	20°	21°	22°	23°	24°	25°	26°	27°	28°	29°	30°	31°	32°	33°	34°	35°	36°	37°	38°	39°	40°	41°	42°	43°	44°	
	0.36997	0.36727	0.37057	0.37057	0.37388	0.37720	0.38053	0.38386	89°	33	66	100	133	166	199	232	265	298								
	0.38386	0.38721	0.39055	0.39391	0.39727	0.40065	0.40403	0.40741	88°	34	67	101	134	168	202	236	269	302								
	0.40403	0.40741	0.41081	0.41421	0.41763	0.42105	0.42447	0.42791	87°	34	68	102	136	170	205	239	273	306								
	0.42447	0.42791	0.43136	0.43481	0.43828	0.44175	0.44523	0.44872	86°	35	69	104	138	173	208	242	277	311								
	0.44523	0.44872	0.45222	0.45573	0.45924	0.46277	0.46631	0.46985	85°	35	70	105	140	176	211	246	281	316								
	0.46985	0.47341	0.47698	0.48055	0.48414	0.48773	0.49133	0.49495	84°	36	71	107	143	179	214	250	286	321								
	0.49495	0.49853	0.50217	0.50587	0.50953	0.51320	0.51688	0.52057	83°	36	72	109	145	182	218	254	291	327								
	0.52057	0.52427	0.52798	0.53171	0.53543	0.53916	0.54289	0.54663	82°	37	74	111	148	185	222	259	296	333								
	0.54663	0.55036	0.55410	0.55783	0.56156	0.56529	0.56902	0.57275	81°	38	75	113	151	189	226	264	302	339								
	0.57275	0.57648	0.58021	0.58394	0.58767	0.59140	0.59513	0.59886	80°	38	77	115	164	192	230	269	307	346								
	0.59886	0.60259	0.60632	0.61005	0.61378	0.61751	0.62124	0.62497	79°	39	78	118	167	196	235	274	313	353								
	0.62497	0.62870	0.63243	0.63616	0.63989	0.64362	0.64735	0.65108	78°	40	80	120	160	200	240	280	320	360								
	0.65108	0.65481	0.65854	0.66227	0.66600	0.66973	0.67346	0.67719	77°	41	82	123	164	205	245	285	327	368								
	0.67719	0.68092	0.68465	0.68838	0.69211	0.69584	0.69957	0.70330	76°	42	83	126	167	209	251	293	334	376								
	0.70330	0.70703	0.71076	0.71449	0.71822	0.72195	0.72568	0.72941	75°	43	86	128	171	214	257	300	342	385								
	0.72941	0.73314	0.73687	0.74060	0.74433	0.74806	0.75179	0.75552	74°	44	88	132	176	220	263	307	351	395								
	0.75552	0.75925	0.76298	0.76671	0.77044	0.77417	0.77790	0.78163	73°	45	90	135	180	225	270	315	360	405								
	0.78163	0.78536	0.78909	0.79282	0.79655	0.80028	0.80401	0.80774	72°	46	92	139	185	231	277	324	370	416								
	0.80774	0.81147	0.81520	0.81893	0.82266	0.82639	0.83012	0.83385	71°	48	95	143	190	238	285	333	380	428								
	0.83385	0.83758	0.84131	0.84504	0.84877	0.85250	0.85623	0.85996	70°	49	98	147	196	245	293	342	391	440								
	0.85996	0.86369	0.86742	0.87115	0.87488	0.87861	0.88234	0.88607	69°	50	101	151	201	252	302	352	402	453								
	0.88607	0.88980	0.89353	0.89726	0.90099	0.90472	0.90845	0.91218	68°	52	104	156	208	260	311	363	415	467								
	0.91218	0.91591	0.91964	0.92337	0.92710	0.93083	0.93456	0.93829	67°	54	107	161	214	268	321	375	429	482								
	0.93829	0.94202	0.94575	0.94948	0.95321	0.95694	0.96067	0.96440	66°	55	111	166	221	277	332	387	442	498								
	0.96440	0.96813	0.97186	0.97559	0.97932	0.98305	0.98678	0.99051	65°	57	114	172	229	286	343	400	457	515								
	0.99051	0.99424	0.99797	1.00170	1.00543	1.00916	1.01289	1.01662		1'	2'	3'	4'	5'	6'	7'	8'	9'								

## NATURAL COTANGENTS



## NATURAL TANGENTS

	Mean Differences																
	0'	10'	20'	30'	40'	50'	60'		1'	2'	3'	4'	5'	6'	7'	8'	9'
45°	1°00000	1°00593	1°01170	1°01761	1°02355	1°02952	1°03553	44°	59	118	178	237	296	355	414	474	533
46°	°03553	°04158	°04766	°05378	°05994	°06613	°07237	43°	61	123	184	246	307	368	430	491	553
47°	°07237	°07864	°08496	°09131	°09770	°10414	°11061	42°	64	127	191	255	319	382	446	510	573
48°	°11061	°11713	°12369	°13029	°13694	°14363	°15037	41°	66	132	199	265	332	397	463	530	596
49°	°15037	°15715	°16398	°17085	°17777	°18474	°19175	40°	69	138	207	276	345	413	482	552	620
50°	1°19175	1°19882	1°20593	1°21310	1°22031	1°22758	1°23490	39°	72	144	216	288	360	431	503	575	647
51°	°23490	°24227	°24969	°25717	°26471	°27230	°27994	38°	75	150	225	300	376	451	526	601	676
52°	°27994	°28764	°29541	°30323	°31110	°31904	°32704	37°	78	157	235	314	392	471	549	628	707
53°	°32704	°33511	°34323	°35142	°35968	°36800	°37638	36°	82	164	247	329	411	493	576	658	740
54°	°37638	°38494	°39336	°40195	°41061	°41934	°42815	35°	86	172	259	345	431	517	603	690	776
55°	1°42815	1°43703	1°44598	1°45501	1°46411	1°47380	1°48256	34°	91	181	272	363	453	544	634	725	816
56°	°48256	°49190	°50133	°51084	°52043	°53010	°53987	33°	96	191	287	382	478	573	669	764	860
57°	°53987	°54972	°55966	°56969	°57981	°59002	°60033	32°	101	201	302	403	504	604	705	806	907
58°	°60033	°61074	°62125	°63185	°64256	°65337	°66428	31°	107	213	320	426	533	639	746	852	959
59°	°66428	°67530	°68643	°69766	°70901	°72047	°73205	30°	113	226	339	451	565	677	790	903	1016
60°	1°73205	1°74377	1°75556	1°76755	1°7796	1°7917	1°8040	29°	12	24	36	48	60	72	84	96	108
61°	°8040	°8165	°8291	°8418	°8546	°8676	°8807	28°	13	25	38	51	64	77	89	102	115
62°	°8807	°8940	°9074	°9210	°9347	°9486	°9626	27°	14	27	41	54	68	82	95	109	122
63°	°9626	°9769	°9912	°10057	°10204	°10353	°10503	26°	15	29	44	58	73	88	102	117	131
64°	°10503	°10655	°10809	°10965	°11123	°11283	°11445	25°	16	31	47	63	79	94	110	126	141

TABLE III.]

## NATURAL TANGENTS

XIII

	60°	61°	62°	63°	64°	65°	66°	67°	68°	69°	70°	71°	72°	73°	74°	75°	76°	77°	78°	79°	80°	81°	82°	83°	84°	85°	86°	87°	88°	89°	90°				
	2.1440	2.1460	2.1475	2.1493	2.1513	2.1536	2.1560	2.1586	2.1613	2.1641	2.1670	2.1700	2.1731	2.1763	2.1796	2.1830	2.1865	2.1901	2.1938	2.1976	2.2015	2.2055	2.2096	2.2138	2.2181	2.2225	2.2270	2.2316	2.2363	2.2411	2.2460	2.2510			
	2.2460	2.2510	2.2560	2.2611	2.2663	2.2715	2.2768	2.2822	2.2877	2.2933	2.2989	2.3046	2.3104	2.3163	2.3222	2.3282	2.3343	2.3404	2.3466	2.3528	2.3591	2.3655	2.3719	2.3784	2.3850	2.3916	2.3983	2.4051	2.4119	2.4188	2.4258	2.4329	2.4401		
	2.4401	2.4473	2.4545	2.4618	2.4692	2.4766	2.4841	2.4917	2.4993	2.5070	2.5148	2.5226	2.5305	2.5384	2.5464	2.5545	2.5626	2.5708	2.5790	2.5873	2.5957	2.6041	2.6126	2.6212	2.6298	2.6385	2.6473	2.6562	2.6651	2.6741	2.6832	2.6924	2.7016		
	2.7016	2.7109	2.7202	2.7296	2.7391	2.7486	2.7582	2.7679	2.7776	2.7874	2.7972	2.8071	2.8171	2.8271	2.8372	2.8474	2.8576	2.8679	2.8782	2.8886	2.8991	2.9096	2.9202	2.9308	2.9415	2.9523	2.9631	2.9740	2.9849	2.9959	3.0069	3.0180	3.0291		
	3.0291	3.0403	3.0516	3.0629	3.0743	3.0858	3.0973	3.1089	3.1205	3.1322	3.1439	3.1557	3.1675	3.1794	3.1913	3.2033	3.2153	3.2274	3.2395	3.2517	3.2639	3.2762	3.2886	3.3010	3.3135	3.3260	3.3386	3.3512	3.3639	3.3766	3.3894	3.4022	3.4151		
	3.4151	3.4280	3.4410	3.4540	3.4671	3.4802	3.4934	3.5066	3.5199	3.5332	3.5466	3.5600	3.5735	3.5870	3.6006	3.6142	3.6279	3.6416	3.6554	3.6692	3.6831	3.6970	3.7110	3.7250	3.7390	3.7531	3.7672	3.7813	3.7955	3.8097	3.8240	3.8383	3.8526	3.8670	
	3.8670	3.8814	3.8958	3.9103	3.9248	3.9393	3.9539	3.9685	3.9831	3.9978	4.0125	4.0272	4.0420	4.0568	4.0716	4.0865	4.1014	4.1163	4.1313	4.1463	4.1613	4.1764	4.1915	4.2066	4.2218	4.2370	4.2522	4.2675	4.2827	4.2980	4.3133	4.3286	4.3440	4.3594	
	4.3594	4.3748	4.3902	4.4057	4.4212	4.4367	4.4523	4.4679	4.4835	4.4992	4.5148	4.5305	4.5462	4.5620	4.5777	4.5935	4.6093	4.6251	4.6410	4.6568	4.6727	4.6886	4.7046	4.7205	4.7365	4.7525	4.7685	4.7846	4.8007	4.8168	4.8329	4.8490	4.8652	4.8814	
	4.8814	4.8976	4.9138	4.9301	4.9464	4.9627	4.9791	4.9955	5.0119	5.0283	5.0448	5.0613	5.0778	5.0943	5.1108	5.1274	5.1439	5.1605	5.1771	5.1937	5.2104	5.2271	5.2438	5.2605	5.2772	5.2940	5.3108	5.3276	5.3445	5.3614	5.3783	5.3953	5.4123	5.4293	
	5.4293	5.4463	5.4634	5.4805	5.4976	5.5147	5.5318	5.5490	5.5662	5.5834	5.6006	5.6178	5.6351	5.6524	5.6697	5.6870	5.7043	5.7216	5.7390	5.7564	5.7738	5.7912	5.8086	5.8261	5.8436	5.8611	5.8786	5.8961	5.9136	5.9312	5.9487	5.9663	5.9839	6.0015	
	6.0015	6.0191	6.0367	6.0544	6.0721	6.0898	6.1075	6.1253	6.1431	6.1609	6.1787	6.1966	6.2145	6.2324	6.2503	6.2682	6.2862	6.3042	6.3222	6.3402	6.3583	6.3763	6.3944	6.4125	6.4306	6.4487	6.4668	6.4849	6.5031	6.5212	6.5394	6.5576	6.5758	6.5940	
	6.5940	6.6123	6.6306	6.6489	6.6672	6.6856	6.7040	6.7224	6.7408	6.7592	6.7776	6.7961	6.8146	6.8331	6.8516	6.8701	6.8886	6.9071	6.9256	6.9442	6.9627	6.9813	6.9999	7.0185	7.0371	7.0557	7.0743	7.0929	7.1115	7.1302	7.1488	7.1675	7.1862	7.2049	
	7.2049	7.2236	7.2423	7.2610	7.2797	7.2985	7.3172	7.3360	7.3548	7.3736	7.3924	7.4112	7.4301	7.4490	7.4679	7.4868	7.5057	7.5246	7.5436	7.5625	7.5815	7.6005	7.6195	7.6385	7.6575	7.6765	7.6956	7.7146	7.7337	7.7528	7.7718	7.7909	7.8100	7.8291	
	7.8291	7.8482	7.8673	7.8864	7.9055	7.9246	7.9437	7.9628	7.9819	8.0010	8.0201	8.0392	8.0583	8.0774	8.0965	8.1156	8.1347	8.1538	8.1729	8.1920	8.2111	8.2302	8.2493	8.2684	8.2875	8.3066	8.3257	8.3448	8.3639	8.3830	8.4021	8.4212	8.4403	8.4594	
	8.4594	8.4785	8.4976	8.5167	8.5358	8.5549	8.5740	8.5931	8.6122	8.6313	8.6504	8.6695	8.6886	8.7077	8.7268	8.7459	8.7650	8.7841	8.8032	8.8223	8.8414	8.8605	8.8796	8.8987	8.9178	8.9369	8.9560	8.9751	8.9942	9.0133	9.0324	9.0515	9.0706	9.0897	
	9.0897	9.1088	9.1279	9.1470	9.1661	9.1852	9.2043	9.2234	9.2425	9.2616	9.2807	9.2998	9.3189	9.3380	9.3571	9.3762	9.3953	9.4144	9.4335	9.4526	9.4717	9.4908	9.5099	9.5290	9.5481	9.5672	9.5863	9.6054	9.6245	9.6436	9.6627	9.6818	9.7009	9.7200	
	9.7200	9.7391	9.7582	9.7773	9.7964	9.8155	9.8346	9.8537	9.8728	9.8919	9.9110	9.9301	9.9492	9.9683	9.9874	10.0065	10.0256	10.0447	10.0638	10.0829	10.1020	10.1211	10.1402	10.1593	10.1784	10.1975	10.2166	10.2357	10.2548	10.2739	10.2930	10.3121	10.3312	10.3503	
	10.3503	10.3694	10.3885	10.4076	10.4267	10.4458	10.4649	10.4840	10.5031	10.5222	10.5413	10.5604	10.5795	10.5986	10.6177	10.6368	10.6559	10.6750	10.6941	10.7132	10.7323	10.7514	10.7705	10.7896	10.8087	10.8278	10.8469	10.8660	10.8851	10.9042	10.9233	10.9424	10.9615	10.9806	
	10.9806	11.0007	11.0208	11.0409	11.0610	11.0811	11.1012	11.1213	11.1414	11.1615	11.1816	11.2017	11.2218	11.2419	11.2620	11.2821	11.3022	11.3223	11.3424	11.3625	11.3826	11.4027	11.4228	11.4429	11.4630	11.4831	11.5032	11.5233	11.5434	11.5635	11.5836	11.6037	11.6238	11.6439	
	11.6439	11.6640	11.6841	11.7042	11.7243	11.7444	11.7645	11.7846	11.8047	11.8248	11.8449	11.8650	11.8851	11.9052	11.9253	11.9454	11.9655	11.9856	12.0057	12.0258	12.0459	12.0660	12.0861	12.1062	12.1263	12.1464	12.1665	12.1866	12.2067	12.2268	12.2469	12.2670	12.2871	12.3072	
	12.3072	12.3273	12.3474	12.3675	12.3876	12.4077	12.4278	12.4479	12.4680	12.4881	12.5082	12.5283	12.5484	12.5685	12.5886	12.6087	12.6288	12.6489	12.6690	12.6891	12.7092	12.7293	12.7494	12.7695	12.7896	12.8097	12.8298	12.8499	12.8700	12.8901	12.9102	12.9303	12.9504	12.9705	
	12.9705	12.9906	13.0107	13.0308	13.0509	13.0710	13.0911	13.1112	13.1313	13.1514	13.1715	13.1916	13.2117	13.2318	13.2519	13.2720	13.2921	13.3122	13.3323	13.3524	13.3725	13.3926	13.4127	13.4328	13.4529	13.4730	13.4931	13.5132	13.5333	13.5534	13.5735	13.5936	13.6137	13.6338	13.6539
	13.6539	13.6740	13.6941	13.7142	13.7343	13.7544	13.7745	13.7946	13.8147	13.8348	13.8549	13.8750	13.8951	13.9152	13.9353	13.9554	13.9755	13.9956	14.0157	14.0358	14.0559	14.0760	14.0961	14.1162	14.1363	14.1564	14.1765	14.1966	14.2167	14.2368	14.2569	14.2770	14.2971	14.3172	
	14.3172	14.3373	14.3574	14.3775	14.3976	14.4177	14.4378	14.4579	14.4780	14.4981	14.5182	14.5383	14.5584	14.5785	14.5986	14.6187	14.6388	14.6589	14.6790	14.6991	14.7192	14.7393	14.7594	14.7795	14.7996	14.8197	14.8398	14.8599	14.8800	14.9001	14.9202	14.9403	14.9604	14.9805	15.0006
	15.0006	15.0207	15.0408	15.0609	15.0810	15.1011	15.1212	15.1413	15.1614	15.1815	15.2016	15.2217	15.2418	15.2619	15.2820	15.3021	15.3222	15.3423	15.3624	15.3825	15.4026	15.4227	15.4428	15.4629	15.4830	15.5031	15.5232	15.5433	15.5634	15.5835	15.6036	15.6237	15.6438	15.6639	15.6840
	15.6840	15.7041	15.7242	15.7443	15.7644	15.7845	15.8046	15.8247	15.8448	15.8649	15.8850	15.9051	15.9252	15.9453	15.9654	15.9855	16.0056	16.0257	16.0458	16.0659	16.0860	16.1061	16.1262	16.1463	16.1664	16.1865	16.2066	16.2267	16.2468	16.2669	16.2870	16.3071	16.3272	16.3473	16.3674
	16.3674	16.3875	16.4076	16.4277	16.4478	16.4679	16.4880	16.5081	16.5282	16.5483	16.5684	16.5885	16.6086	16.6287	16.6488	16.6689	16.6890	16.7091	16.7292	16.7493	16.7694	16.7895	16.8096	16.8297	16.8498	16.8699	16.8900	16.9101	16.9302	16.9503	16.9704	16.9905	17.0106	17.0307	17.0508
	17.0508	17.0709	17.0910	17.1111	17.1312	17.1513	17.1714	17.1915	17.2116	17.2317	17.2518	17.2719	17.2920	17.3121	17.3322	17.3523	17.3724	17.3925	17.4126	17.4327	17.4528	17.4729	17.4930	17.5131	17.5332	17.5533	17.5734	17.5935	17.6136	17.6337	17.6538	17.6739	17.6940	17.7141	17.7342
	17.7342	17.7543	17.7744	17.7945	17.8146	17.8347	17.8548	17.8749	17.8950	17.9151	17.9352	17.9553	17.9754	17.9955	18.0156	18.0357	18.0558	18.0759	18.0960	18.1161	18.1362	18.1563	18.1764	18.1965	18.2166	18.2367	18.2568	18.2769	18.2970	18.3171	18.3372	18.3573	18.3774	18.3975	18.4176
	18.4176	18.4377	18.4578	18.4779	18.4980	18.5181	18.5382	18.5583	18.5784	18.5985	18.6186	18.6387	18.6588	18.6789	18.6990	18.7191	18.7392	18.7593	18.7794	18.7995	18.8196	18.8397	18.8598	18.8799	18.8999	18.9200	18.9401	18.9602	18.9803	18.9999	19.0199	19.0399	19.0599	19.0799	19.



TABLE IV.]

## LOGARITHMIC SINES

X

	60'	50'	40'	30'	20'	10'	0'		1	2'	3'	4'	5'	6'	7'	8'	9'
20°	9°53405	9°53751	9°54093	9°54439	9°54769	9°55102	9°55433	69°	34	68	101	135	169	203	237	270	304
21°	55433	55751	56085	56408	56727	57044	57358	68°	32	64	96	128	161	193	225	257	289
22°	57358	57669	57978	58284	58588	58889	59188	67°	31	61	92	122	153	183	214	244	275
23°	59188	59484	59778	60070	60359	60646	60931	66°	29	58	87	116	146	174	204	233	262
24°	60931	61214	61494	61773	62049	62323	62595	65°	28	56	83	111	139	166	195	222	250
25°	62595	62865	63133	63398	63662	63924	64184	64°	27	53	80	106	133	159	186	212	239
26°	64184	64442	64698	64953	65205	65456	65705	63°	25	51	76	102	127	152	178	203	229
27°	65705	65952	66197	66441	66682	66922	67161	62°	24	49	73	97	122	146	170	194	219
28°	67161	67398	67633	67866	68098	68328	68557	61°	23	47	70	93	117	140	163	186	210
29°	68557	68784	69010	69234	69456	69677	69897	60°	22	45	67	89	112	134	156	179	201
30°	69897	70115	70332	70547	70761	70973	71184	59°	22	43	65	86	107	129	150	172	193
31°	71184	71393	71602	71809	72014	72218	72421	58°	21	41	62	82	103	124	144	165	185
32°	72421	72623	72823	73023	73219	73416	73611	57°	20	40	59	79	99	119	139	159	178
33°	73611	73805	73997	74189	74379	74568	74756	56°	19	38	57	76	96	116	134	153	172
34°	74756	74943	75128	75313	75496	75678	75859	55°	18	37	55	74	92	110	129	147	165
35°	75859	76039	76218	76395	76572	76747	76922	54°	18	35	53	71	89	106	124	142	159
36°	76922	77095	77268	77439	77609	77778	77946	53°	17	34	51	68	86	103	120	137	154
37°	77946	78113	78280	78445	78609	78772	78934	52°	17	33	50	66	83	99	116	132	149
38°	78934	79095	79256	79415	79573	79731	79887	51°	16	32	48	64	80	95	112	127	143
39°	79887	80043	80197	80351	80504	80656	80807	50°	15	31	46	62	77	92	107	123	138
40°	80807	80957	81106	81254	81402	81549	81694	49°	15	30	44	59	74	89	104	118	133
41°	81694	81839	81983	82126	82269	82410	82551	48°	14	29	43	57	72	86	100	114	129
42°	82551	82691	82830	82968	83106	83242	83378	47°	14	28	41	55	69	83	97	110	124
43°	83378	83513	83648	83781	83914	84046	84177	46°	13	27	40	53	67	80	93	106	120
44°	84177	84309	84437	84566	84694	84822	84949	45°	13	26	38	51	64	77	90	102	115

## LOGARITHMIC COSINES

## LOGARITHMIC SINES

	0'	10'	20'	30'	40'	50'	60'		1'	2'	3'	4'	5'	6'	7'	8'	9'
45°	9.84949	9.85074	9.85200	9.85324	9.85448	9.85571	9.85693	44°	12	25	37	50	62	74	87	99	112
46°	9.85693	9.85815	9.85936	9.86056	9.86176	9.86294	9.86413	43°	12	24	36	48	60	72	84	96	108
47°	9.86413	9.86530	9.86647	9.86763	9.86879	9.86993	9.87107	42°	12	23	35	46	58	70	81	93	104
48°	9.87107	9.87221	9.87334	9.87446	9.87557	9.87668	9.87778	41°	11	22	34	45	56	67	78	89	100
49°	9.87778	9.87887	9.87996	9.88105	9.88212	9.88319	9.88425	40°	11	22	32	43	54	65	76	86	97
50°	9.88425	9.88531	9.88636	9.88741	9.88844	9.88948	9.89050	39°	10	21	31	42	52	62	73	83	94
51°	9.89050	9.89152	9.89254	9.89354	9.89455	9.89553	9.89653	38°	10	20	30	40	50	60	70	80	90
52°	9.89653	9.89752	9.89849	9.89947	9.90043	9.90139	9.90235	37°	10	19	29	39	49	58	68	78	87
53°	9.90235	9.90330	9.90424	9.90518	9.90611	9.90704	9.90796	36°	9	19	28	37	47	56	65	74	84
54°	9.90796	9.90887	9.90978	9.91069	9.91158	9.91241	9.91326	35°	9	18	27	36	45	54	63	72	81
55°	9.91326	9.91425	9.91512	9.91599	9.91686	9.91772	9.91857	34°	9	17	26	35	44	52	61	70	78
56°	9.91857	9.91942	9.92027	9.92111	9.92194	9.92277	9.92359	33°	8	17	25	34	42	50	59	67	76
57°	9.92359	9.92441	9.92522	9.92603	9.92683	9.92763	9.92842	32°	8	16	24	32	41	49	57	65	73
58°	9.92842	9.92921	9.92999	9.93077	9.93154	9.93230	9.93307	31°	8	16	23	31	39	47	55	62	70
59°	9.93307	9.93382	9.93457	9.93532	9.93606	9.93680	9.93753	30°	8	15	23	30	37	45	53	60	67
60°	9.93753	9.93826	9.93898	9.93970	9.94041	9.94112	9.94183	29°	7	14	22	29	36	43	50	57	64
61°	9.94183	9.94252	9.94321	9.94390	9.94458	9.94526	9.94593	28°	7	14	21	27	34	41	48	55	62
62°	9.94593	9.94660	9.94727	9.94793	9.94858	9.94923	9.94988	27°	7	13	20	26	33	40	46	53	59
63°	9.94988	9.95052	9.95116	9.95179	9.95242	9.95304	9.95366	26°	6	13	19	25	32	38	44	50	57
64°	9.95366	9.95427	9.95488	9.95549	9.95609	9.95668	9.95728	25°	6	12	18	24	30	36	42	48	54

TABLE IV

## LOGARITHMIC SINES

XVII

[illegible]

## LOGARITHMIC COSINES

**TABLE V**  
**LOGARITHMIC TANGENTS**

[illegible]

TABLE V ]

## LOGARITHMIC TANGENTS

XIX

	60'	50'	40'	30'	20'	10'	0'	1	2	3	4	5	6	7	8	9	
20°	9.56107	9.56498	9.56887	9.57274	9.57658	9.58039	9.58418	69°	39	77	116	154	193	231	270	308	347
21°	9.56418	9.56794	9.57168	9.57540	9.57909	9.58276	9.58641	68°	37	74	111	148	185	222	259	296	333
22°	9.56641	9.57004	9.57364	9.57722	9.58079	9.58433	9.58785	67°	36	72	107	143	179	214	250	286	322
23°	9.56785	9.57135	9.57483	9.57830	9.58175	9.58518	9.58858	66°	35	69	104	138	173	208	242	277	311
24°	9.56858	9.57197	9.57535	9.57870	9.58204	9.58537	9.58867	65°	34	67	101	134	168	201	235	268	302
25°	9.56867	9.57196	9.57524	9.57850	9.58174	9.58497	9.58818	64°	33	65	98	130	163	195	228	260	293
26°	9.56818	9.57138	9.57457	9.57774	9.58089	9.58394	9.58691	63°	32	63	95	126	158	190	221	253	284
27°	9.56717	9.57023	9.57329	9.57635	9.57939	9.58242	9.58543	62°	31	62	92	123	154	185	216	246	277
28°	9.56567	9.56872	9.57175	9.57476	9.57777	9.58077	9.58375	61°	30	60	90	120	151	181	211	241	271
29°	9.56435	9.56738	9.57039	9.57339	9.57638	9.57935	9.58231	60°	29	59	88	118	147	177	206	236	265
30°	9.56144	9.56435	9.56725	9.57015	9.57303	9.57591	9.57877	59°	29	58	87	116	144	173	202	231	260
31°	9.55777	9.56063	9.56348	9.56632	9.56915	9.57197	9.57478	58°	28	57	85	113	142	170	198	227	255
32°	9.55579	9.55860	9.56140	9.56419	9.56697	9.56975	9.57252	57°	28	56	84	112	139	167	195	223	251
33°	9.55252	9.55528	9.55803	9.56078	9.56352	9.56626	9.56899	56°	28	55	83	110	137	165	192	220	247
34°	9.54899	9.55171	9.55442	9.55713	9.55984	9.56254	9.56523	55°	27	54	81	108	136	162	190	217	244
35°	9.54523	9.54791	9.55059	9.55327	9.55594	9.55860	9.56126	54°	27	54	80	107	134	160	188	214	241
36°	9.54126	9.54392	9.54656	9.54921	9.55185	9.55449	9.55711	53°	26	53	79	106	132	158	185	212	238
37°	9.53711	9.53974	9.54236	9.54498	9.54759	9.55020	9.55281	52°	26	52	78	105	131	157	183	209	236
38°	9.53281	9.53541	9.53801	9.54061	9.54320	9.54578	9.54837	51°	26	52	77	104	130	156	182	208	234
39°	9.52837	9.53095	9.53353	9.53610	9.53868	9.54125	9.54381	50°	26	52	77	103	129	155	180	206	232
40°	9.52381	9.52638	9.52894	9.53150	9.53406	9.53661	9.53916	49°	26	51	77	102	128	154	179	205	230
41°	9.51916	9.52171	9.52426	9.52681	9.52935	9.53190	9.53444	48°	25	51	76	102	127	153	178	204	229
42°	9.51444	9.51698	9.51952	9.52205	9.52458	9.52712	9.52966	47°	25	51	76	101	127	152	177	203	228
43°	9.50966	9.51219	9.51472	9.51725	9.51978	9.52231	9.52484	46°	25	51	76	101	127	152	177	203	228
44°	9.50481	9.50737	9.50989	9.51242	9.51495	9.51747	9.52000	45°	25	51	76	101	127	152	177	203	228

## LOGARITHMIC COTANGENTS



## LOGARITHMIC TANGENTS

	Mean Differences																
	1'	2'	3'	4'	5'	6'	7'	8'	9'								
45°	10.00000	10.00253	10.00505	10.00758	10.01011	10.01263	10.01516	44°	25	51	76	101	127	153	177	202	228
46°	.01516	.01769	.02022	.02275	.02528	.02781	.03034	43°	25	51	76	101	127	152	177	202	228
47°	.03034	.03288	.03541	.03795	.04048	.04302	.04556	42°	25	51	76	101	127	152	177	203	228
48°	.04556	.04810	.05065	.05319	.05574	.05829	.06084	41°	25	51	76	102	127	153	178	204	229
49°	.06084	.06339	.06594	.06850	.07106	.07362	.07619	40°	26	51	77	103	128	154	179	205	230
50°	10.07619	10.07875	10.08132	10.08390	10.08647	10.08905	10.09163	39°	26	52	77	103	129	155	180	206	232
51°	.09163	.09422	.09680	.09939	.10199	.10459	.10719	38°	26	52	78	104	130	156	182	208	234
52°	.10719	.10980	.11241	.11502	.11764	.12026	.12289	37°	26	52	78	105	131	157	183	209	236
53°	.12289	.12552	.12815	.13079	.13344	.13608	.13874	36°	26	53	79	106	132	158	185	212	238
54°	.13874	.14140	.14406	.14673	.14941	.15209	.15477	35°	27	54	80	107	134	160	186	214	241
55°	10.15477	10.15746	10.16016	10.16287	10.16558	10.16829	10.17101	34°	27	54	81	108	136	162	190	217	244
56°	.17101	.17374	.17648	.17922	.18197	.18472	.18748	33°	28	55	83	110	137	165	192	220	247
57°	.18748	.19025	.19303	.19581	.19860	.20140	.20421	32°	28	56	84	112	139	167	195	223	251
58°	.20421	.20703	.20985	.21268	.21552	.21837	.22123	31°	28	57	85	113	142	170	198	227	255
59°	.22123	.22409	.22697	.22986	.23275	.23565	.23856	30°	29	58	87	116	144	173	202	231	260
60°	10.23856	10.24148	10.24442	10.24736	10.25031	10.25327	10.25625	29°	29	59	88	118	147	177	206	236	265
61°	.25625	.25923	.26223	.26524	.26825	.27128	.27433	28°	30	60	90	120	151	181	211	241	271
62°	.27433	.27738	.28045	.28352	.28661	.28972	.29283	27°	31	62	92	123	154	185	216	246	277
63°	.29283	.29596	.29911	.30226	.30543	.30862	.31182	26°	32	63	95	126	158	190	221	253	284
64°	.31182	.31503	.31826	.32150	.32476	.32804	.33133	25°	33	65	98	130	163	195	228	260	293

TABLE V ]

## LOGARITHMIC TANGENTS

XXI

	65°	66°	67°	68°	69°	70°	71°	72°	73°	74°	75°	76°	77°	78°	79°	80°	81°	82°	83°	84°	85°	86°	87°	88°	89°	90°																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																															
	10°38133	10°38468	10°38796	10°39130	10°39465	10°39803	10°40143	24°	23°	22°	21°	20°	19°	18°	17°	16°	15°	14°	13°	12°	11°	10°	9°	8°	7°	6°	5°	4°	3°	2°	1°	0°																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																									
	35143	35483	35825	36170	36516	36865	37215	37567	37921	38278	38636	38996	39359	39724	40091	40460	40832	41206	41582	41961	42342	42726	43113	43502	43893	44285	44678	45072	45468	45864	46261	46659	47058	47458	47859	48261	48664	49068	49473	49879	50286	50694	51103	51513	51924	52336	52749	53163	53578	53994	54411	54829	55248	55668	56089	56511	56934	57358	57783	58209	58636	59064	59493	59923	60354	60786	61219	61653	62088	62524	62961	63399	63838	64278	64719	65161	65604	66048	66493	66939	67386	67834	68283	68733	69184	69636	70089	70543	70998	71454	71911	72369	72828	73288	73749	74211	74674	75138	75603	76069	76536	77004	77473	77943	78414	78886	79359	79833	80308	80784	81261	81739	82218	82698	83179	83661	84144	84628	85113	85599	86086	86574	87063	87553	88044	88536	89029	89523	90018	90514	91011	91509	92008	92508	93009	93511	94014	94518	95023	95529	96036	96544	97053	97563	98074	98586	99099	99613	100148	100664	101181	101699	102218	102738	103259	103781	104304	104828	105353	105879	106406	106934	107463	107993	108524	109056	109589	110123	110658	111194	111731	112269	112808	113348	113889	114431	114974	115518	116063	116609	117156	117704	118253	118803	119354	119906	120459	121013	121568	122124	122681	123239	123798	124358	124919	125481	126044	126608	127173	127739	128306	128874	129443	130013	130584	131156	131729	132303	132878	133454	134031	134609	135188	135768	136349	136931	137514	138098	138683	139269	139856	140444	141033	141623	142214	142806	143399	143993	144588	145184	145781	146379	146978	147578	148179	148781	149384	150000	150617	151235	151854	152474	153095	153717	154340	154964	155589	156215	156842	157470	158099	158729	159360	160000	160641	161283	161926	162570	163215	163861	164508	165156	165805	166455	167106	167758	168411	169065	169720	170376	171033	171691	172350	173010	173671	174333	175000	175668	176337	177007	177678	178350	179023	179697	180372	181048	181725	182403	183082	183762	184443	185125	185808	186492	187177	187863	188550	189238	189927	190617	191308	192000	192693	193387	194082	194778	195475	196173	196872	197572	198273	198975	199678	200382	201087	201793	202500	203208	203917	204627	205338	206050	206763	207477	208192	208908	209625	210343	211062	211782	212503	213225	213948	214672	215397	216123	216850	217578	218307	219037	219768	220500	221233	221967	222702	223438	224175	224913	225652	226392	227133	227875	228618	229362	230107	230853	231600	232348	233097	233847	234598	235350	236103	236857	237612	238368	239125	239883	240642	241402	242163	242925	243688	244452	245217	245983	246750	247518	248287	249057	249828	250600	251373	252147	252922	253698	254475	255253	256032	256812	257593	258375	259158	259942	260727	261513	262300	263088	263877	264667	265458	266250	267043	267837	268632	269428	270225	271023	271822	272622	273423	274225	275028	275832	276637	277443	278250	279058	279867	280677	281488	282299	283112	283926	284741	285557	286374	287192	288011	288831	289652	290474	291297	292121	292946	293772	294600	295428	296257	297087	297918	298750	299583	300417	301252	302088	302925	303763	304602	305442	306283	307125	307968	308812	309657	310503	311350	312198	313047	313897	314748	315600	316453	317307	318162	319018	319875	320733	321592	322452	323313	324175	325038	325902	326767	327633	328500	329368	330237	331107	331978	332850	333723	334597	335472	336348	337225	338103	338982	339862	340743	341625	342508	343392	344277	345163	346050	346938	347827	348717	349608	350500	351393	352287	353182	354078	354975	355873	356772	357672	358573	359475	360378	361282	362187	363093	364000	364908	365817	366727	367638	368550	369463	370377	371292	372208	373125	374043	374962	375882	376803	377725	378648	379572	380497	381423	382350	383278	384207	385137	386068	386999	387932	388866	389801	390737	391674	392612	393551	394491	395432	396374	397317	398261	399206	400152	401100	402048	402998	403948	404899	405851	406804	407758	408713	409669	410626	411584	412543	413503	414464	415426	416389	417353	418318	419284	420251	421219	422188	423158	424129	425101	426074	427048	428023	428999	429976	430954	431933	432913	433894	434876	435859	436843	437828	438814	439801	440789	441778	442768	443759	444751	445744	446738	447733	448729	449726	450724	451723	452723	453724	454726	455729	456733	457738	458743	459749	460756	461764	462772	463781	464791	465801	466812	467824	468836	469849	470863	471877	472892	473908	474924	475941	476959	477977	478996	479016	480036	481057	482078	483099	484121	485144	486168	487193	488218	489243	490269	491295	492322	493350	494378	495407	496437	497467	498498	499529	500561	501594	502627	503661	504696	505731	506767	507803	508840	509878	510916	511955	512994	514034	515075	516116	517158	518200	519243	520286	521330	522375	523420	524466	525512	526559	527606	528654	529702	530751	531800	532850	533900	534951	536002	537054	538106	539159	540212	541266	542320	543375	544430	545485	546541	547597	548654	549711	550769	551827	552886	553945	555005	556065	557126	558187	559248	560310	561372	562435	563498	564562	565626	566691	567756	568821	569887	570953	572020	573087	574155	575223	576292	577361	578431	579501	580572	581643	582714	583786	584858	585931	587004	588078	589152	590227	591302	592378	593454	594531	595608	596686	597764	598843	599922	601002	602082	603163	604244	605325	606407	607489	608571	609654	610737	611821	612905	613990	615075	616160	617246	618332	619419	620506	621594	622682	623771	624860	625950	627040	628131	629222	630314	631406	632500	633594	634689	635784	636880	637976	639073	640171	641270	642369	643469	644569	645670	646771	647873	648975	650078	651181	652285	653389	654494	655600	656706	657813	658920	660028	661136	662245	663354	664464	665574	666685	667796	668907	670019	671131	672244	673357	674471	675586	676701	677816	678932	680048	681165	682282	683399	684517	685635	686754	687873	688992	690112	691232	692353	693474	694595	695717	696839	697962	699085	700209	701333	702458	703583	704708	705834	706960	708087	709214	710342	711470	712599	713728	714858	715988	717118	718249	719380	720512	721644	722776	723909	725042	726176	727310	728444	729579	730714	731849	732984	734120	735256	736392	737528	738665	739802	740939	742076	743214	744352	745490	746628	747766	748905	750044	751183	752323	753463	754603	755743	756884	758025	759166	760307	761448	762590	763731	764873	766015	767157	768300	769442	770585	771728	772871	774014	775158	776302	777446	778590	779734	780879	782023	783168	784313	785458	786603	787748	788893	790038	791183	792328	793473	794618	795763	796908	798053	799200	800346	801492	802638	803784	804930	806076	807223	808369	809516	810662	811809	812956	814103	815250	816397	817544	818691	819838	820985	822132	823279	824426	825573	826720	827867	829014	830161	831308	832455	833602	834749	835896	837043	838190	839337	840484	841631	842778	843925	845072	846219	847366	848513	849660	850807	851954	853101	854248	855395	856542	857689	858836	859983	861130	862277	863424	864571	865718	866865	868012	869159	870306	871453	872600	873747	874894	876041	877188	878335	879482	880629	881776	882923	884070	885217	886364	887511	888658	889805	890952	892099	893246	894393	895540	896687	897834	898981	900128	901275	902422	903569	904716	905863	907010	908157	909304	910451	911598	912745	913892	915039	916186	917333	918480	919627	920774	921921	923068	924215	925362	926509	927656	928803	929950	931097	932244	933391	934538	935685	936832	937979	939126	940273	941420	942567	943714	944861	946008	947155	948302	949449	950596	951743	952890	954037	955184	956331	957478	958625	959772	960919	962066	963213	964360	965507	966654	967801	968948	970095	971242	972389</

## SOME USEFUL CONSTANTS

One radian =  $57^{\circ} 17' 45''$  nearly = 206265" ;

$$\log 206265 = 5.3144255.$$

$$\pi = 3.14159265\ldots \qquad \frac{1}{\pi} = 0.31830989.$$

$$\sqrt{2} = 1.4142135\ldots \qquad \sqrt{3} = 1.7320508\ldots$$

$$\sqrt{5} = 2.2360679\ldots \qquad \sqrt{6} = 2.4494897\ldots$$

$$\sqrt{7} = 2.6457513\ldots \qquad \sqrt{8} = 2.8284271\ldots$$

$$\sqrt{10} = 3.1622776\ldots$$

## SOME USEFUL LOGARITHMS

$$\log 2 = .30103 \qquad \log 3 = .47712$$

$$\log 5 = .69897 \qquad \log 7 = .84510$$















